ELSEVIER

Available online at www.sciencedirect.com

ScienceDirect

Nuclear Physics B 831 [FS] (2010) 359–407

NUCLEAR PHYSICS B

www.elsevier.com/locate/nuclphysb

Correlation functions of the integrable higher-spin XXX and XXZ spin chains through the fusion method

Tetsuo Deguchi [a,*], Chihiro Matsui [b,c]

[a] *Department of Physics, Graduate School of Humanities and Sciences, Ochanomizu University, 2-1-1 Ohtsuka, Bunkyo-ku, Tokyo 112-8610, Japan*

[b] *Department of Physics, Graduate School of Science, the University of Tokyo, 7-3-1 Hongo, Bunkyo-ku, Tokyo 113-0033, Japan*

[c] *CREST, JST, 4-1-8 Honcho Kawaguchi, Saitama 332-0012, Japan*

Received 23 July 2009; accepted 21 December 2009

Available online 4 January 2010

Abstract

For the integrable higher-spin XXX and XXZ spin chains we present multiple-integral representations for the correlation function of an arbitrary product of Hermitian elementary matrices in the massless ground state. We give a formula expressing it by a single term of multiple integrals. In particular, we explicitly derive the emptiness formation probability (EFP). We assume $2s$-strings for the ground-state solution of the Bethe-ansatz equations for the spin-s XXZ chain, and solve the integral equations for the spin-s Gaudin matrix. In terms of the XXZ coupling Δ we define ζ by $\Delta = \cos\zeta$, and put it in a region $0 \leqslant \zeta < \pi/2s$ of the gapless regime: $-1 < \Delta \leqslant 1$ $(0 \leqslant \zeta < \pi)$, where $\Delta = 1$ $(\zeta = 0)$ corresponds to the antiferromagnetic point. We calculate the zero-temperature correlation functions by the algebraic Bethe-ansatz, introducing the Hermitian elementary matrices in the massless regime, and taking advantage of the fusion construction of the R-matrix of the higher-spin representations of the affine quantum group.

1. Introduction

The correlation functions of the spin-1/2 XXZ spin chain have been studied extensively through the algebraic Bethe-ansatz during the last decade [1–6]. The multiple-integral representations of the correlation functions for the infinite lattice at zero temperature first derived through

* Corresponding author.
E-mail addresses: deguchi@phys.ocha.ac.jp (T. Deguchi), matsui@spin.phys.s.u-tokyo.ac.jp (C. Matsui).

doi:10.1016/j.nuclphysb.2009.12.030

the affine quantum-group symmetry [7,8] and also by solving the q-KZ equations [9,10] have been rederived and then generalized into those for the finite-size lattice under nonzero magnetic field. They are also extended into those at finite temperatures [11]. Furthermore, the asymptotic expansion of a correlation function has been systematically discussed [12]. Thus, the exact study of the correlation functions of the XXZ spin chain should be not only very fruitful but also quite fundamental in the mathematical physics of integrable models.

Recently, the correlation functions and form factors of the integrable higher-spin XXX spin chains and the form factors of the integrable higher-spin XXZ spin chains have been derived by the algebraic Bethe-ansatz method [13–15]. In the spin-$1/2$ XXZ chain the Hamiltonian under the periodic boundary conditions is given by

$$\mathcal{H}_{\mathrm{XXZ}} = \frac{1}{2}\sum_{j=1}^{L}\bigl(\sigma_j^X\sigma_{j+1}^X + \sigma_j^Y\sigma_{j+1}^Y + \Delta\sigma_j^Z\sigma_{j+1}^Z\bigr). \tag{1.1}$$

Here σ_j^a $(a = X, Y, Z)$ are the Pauli matrices defined on the jth site and Δ denotes the XXZ coupling. We define parameter q by

$$\Delta = \bigl(q + q^{-1}\bigr)/2. \tag{1.2}$$

We define η and ζ by $q = \exp\eta$ and $\eta = i\zeta$, respectively. We thus have $\Delta = \cos\zeta$. In the massless regime: $-1 < \Delta \leqslant 1$, we have $0 \leqslant \zeta < \pi$ for the spin-$1/2$ XXZ spin chain (1.1). At $\Delta = 1$ (i.e. $q = 1$), the Hamiltonian (1.1) corresponds to the antiferromagnetic Heisenberg (XXX) chain. The solvable higher-spin generalizations of the XXX and XXZ spin chains have been studied by the fusion method in several references [16–23]. The spin-s XXZ Hamiltonian is derived from the spin-s fusion transfer matrix (see also Section 2.6). For instance, the Hamiltonian of the integrable spin-1 XXX spin chain is given by

$$\mathcal{H}_{\mathrm{XXX}}^{(2)} = \frac{1}{2}\sum_{j=1}^{N_s}\bigl(\vec{S}_j\cdot\vec{S}_{j+1} - (\vec{S}_j\cdot\vec{S}_{j+1})^2\bigr). \tag{1.3}$$

Here $\vec{S}_j$ denotes the spin-1 spin-angular momentum operator acting on the jth site among the N_s lattice sites of the spin-s chain. For the general spin-s case, the integrable spin-s XXX and XXZ Hamiltonians denoted $\mathcal{H}_{\mathrm{XXX}}^{(2s)}$ and $\mathcal{H}_{\mathrm{XXZ}}^{(2s)}$, respectively, can also be derived systematically.

The correlation functions of integrable higher-spin XXX and XXZ spin chains are associated with various topics of mathematical physics. For the integrable spin-1 XXZ spin chain correlation functions have been derived by the method of q-vertex operators through some novel results of the representation theory of the quantum algebras [24–28]. They should be closely related to the higher-spin solutions of the quantum Knizhnik–Zamolodchikov equations [10]. For the fusion eight-vertex models, correlation functions have been discussed by an algebraic method [29]. Moreover, the partition function of the six-vertex model under domain wall boundary conditions have been extended into the higher-spin case [30].

In a massless region $0 \leqslant \zeta < \pi/2s$, the low-lying excitation spectrum at zero temperature of the integrable spin-s XXZ chain should correspond to the level-k $SU(2)$ WZWN model with $k = 2s$. By assuming the string hypothesis it is conjectured that the ground state of the integrable spin-s XXX Hamiltonian is given by $N_s/2$ sets of $2s$-strings [31]. It has also been extended into the XXZ case [32]. The ground-state solution of $2s$-strings is derived for the spin-s XXX chain through the zero-temperature limit of the thermal Bethe-ansatz [18]. The low-lying excitation spectrum is discussed in terms of spinons for the spin-s XXX and XXZ spin chains [31,32]. Numerically it was shown that the finite-size corrections to the ground-state energy of the integrable spin-s XXX chain are consistent with the conformal field theory (CFT) with $c = 3s/(s+1)$ [33–36]. Here c denotes the central charge of the CFT. It is also the case with the integrable spin-s XXZ chain in the region $0 \leqslant \zeta < \pi/2s$ [37–39]. The results are consistent with the conjecture that the ground state of the integrable spin-s XXZ chain with $0 \leqslant \zeta < \pi/2s$ is given by $N_s/2$ sets of $2s$-strings [22,32,37–42]. Furthermore, it was shown analytically that the low-lying excitation spectrum of the integrable spin-s XXZ chain in the region $0 \leqslant \zeta < \pi/2s$ is consistent with the CFT of $c = 3s/(s+1)$ [41,42]. In fact, the low-lying excitation spectrum of spinons for the spin-s XXX chain is described in terms of the level-k $SU(2)$ WZWN model with $k = 2s$ [43].

In the paper we calculate zero-temperature correlation functions for the integrable higher-spin XXZ spin chains by the algebraic Bethe-ansatz method. For a given product of elementary matrices we present the multiple-integral representations of the correlation function in the region $0 \leqslant \zeta < \pi/2s$ of the massless regime near the antiferromagnetic point ($\zeta = 0$). For an illustration, we derive the multiple-integral representations of the emptiness formation probability (EFP) of the spin-s XXZ spin chain, explicitly. Here the spin s is given by an arbitrary positive integer or half-integer. Assuming the conjecture that the ground-state solution of the Bethe-ansatz equations is given by $2s$-strings for the regime of ζ, we derive the spin-s EFP for a finite chain and then take the thermodynamic limit. We solve the integral equations associated with the spin-s Gaudin matrix for $0 \leqslant \zeta < \pi/2s$, and express the diagonal elements in terms of the density of strings. Here we remark that the integral equations associated with the spin-s Gaudin matrix have not been explicitly solved, yet, even for the case of the integrable higher-spin XXX spin chains [13]. We also calculate the spin-s EFP for the homogeneous chain where all inhomogeneous parameters ξ_p are given by zero. Here we shall introduce inhomogeneous parameters ξ_p for $p = 1, 2, \ldots, N_s$, in Section 2.4. Furthermore, we take advantage of the fusion construction of the spin-s R-matrix in the algebraic Bethe-ansatz derivation of the correlation functions [15].

Given the spin-s XXZ spin chain on the N_s lattice sites, we define L by $L = 2sN_s$ and consider the spin-$1/2$ XXZ spin chain on the L sites with inhomogeneous parameters w_j for $j = 1, 2, \ldots, L$. In the fusion method we express any given spin-s local operator as a sum of products of operator-valued elements of the spin-$1/2$ monodromy matrix in the limit of sending inhomogeneous parameters w_j to sets of complete $2s$-strings as shown in Ref. [15]. Here, we apply the spin-$1/2$ formula of the quantum inverse scattering problem [4], which is valid at least for generic inhomogeneous parameters. Therefore, sending inhomogeneous parameters w_j into complete $2s$-strings, we can evaluate the vacuum expectation values or the form factors of spin-s local operators which are expressed in terms of the spin-$1/2$ monodromy matrix elements with generic inhomogeneous parameters w_j. Here, the rapidities of the ground state satisfy the Bethe-ansatz equations with inhomogeneous parameters w_j. We assume in the paper that the Bethe roots are continuous with respect to inhomogeneous parameters w_j, in particular, in the limit of sending w_j to complete $2s$-strings.

We can construct higher-spin transfer matrices by the fusion method [22,23]. Here we recall that the spin-$1/2$ XXZ Hamiltonian (1.1) is derived from the logarithmic derivative of the row-to-row transfer matrix of the six-vertex model. We call it the spin-$1/2$ transfer matrix and denote it by $t^{(1,1)}(\lambda)$. Let us express by $V^{(\ell)}$ an $(\ell+1)$-dimensional vector space. We denote by $T^{(\ell,2s)}(\lambda)$ the spin-$\ell/2$ monodromy matrix acting on the tensor product of the auxiliary space $V^{(\ell)}$ and the N_sth tensor product of the quantum spaces, $(V^{(2s)})^{\otimes N_s}$. We call it of type $(\ell, (2s)^{\otimes N_s})$, which we express $(\ell, 2s)$ in the superscript. Taking the trace of the spin-$\ell/2$ monodromy matrix $T^{(\ell,2s)}(\lambda)$ over the auxiliary space $V^{(\ell)}$, we define the spin-$\ell/2$ transfer matrix, $t^{(\ell,2s)}(\lambda)$. For $\ell = 2s$, we have the spin-s transfer matrix $t^{(2s,2s)}(\lambda)$, and we derive the spin-s Hamiltonian from its logarithmic derivative.

We construct the ground state $|\psi_g^{(2s)}\rangle$ of the spin-s XXZ Hamiltonian by the B operators of the 2-by-2 monodromy matrix $T^{(1,2s)}(\lambda)$. As shown by Babujian, the spin-s transfer matrix $t^{(2s,2s)}(\lambda)$ commutes with the spin-1/2 transfer matrix $t^{(1,2s)}(\lambda)$ due to the Yang–Baxter relations, and hence they have eigenvectors in common [18]. The ground state $|\psi_g^{(2s)}\rangle$ of the spin-s XXZ spin chain is originally an eigenvector of the spin-s transfer matrix $t^{(2s,2s)}(\lambda)$, and consequently it is also an eigenvector of the spin-1/2 transfer matrix $t^{(1,2s)}(\lambda)$. Therefore, the ground state $|\psi_g^{(2s)}\rangle$ of the spin-s XXZ spin chain can be constructed by applying the B operators of the 2-by-2 monodromy matrix $T^{(1,2s)}(\lambda)$ to the vacuum.

We can show that the fusion R-matrix corresponds to the R-matrix of the affine quantum group $U_q(\widehat{sl_2})$. We recall that by the fusion method, we can construct the R-matrix acting on the tensor product $V^{(\ell)} \otimes V^{(2s)}$ [16–23]. We denote it by $R^{(\ell,2s)}$. In the affine quantum group, the R-matrix is defined as the intertwiner of the tensor product of two representations V and W [44–46]. Due to the conditions of the intertwiner the R-matrix of the affine quantum group is determined uniquely up to a scalar factor [47], which we denote by $R_{V,W}$. Therefore, showing that the fusion R-matrix satisfies all the conditions of the intertwiner, we prove that the fusion R-matrix coincides with the R-matrix of the quantum group, $R_{V,W}$. Consequently, for $\ell = 2s$ the fusion R-matrix, $R^{(2s,2s)}(\lambda)$, becomes the permutation operator when spectral parameter λ is given by zero. This property of the R-matrix plays a central role in the derivation of the integrable spin-s Hamiltonian. It is also fundamental in the inverse scattering problem in the spin-s case [48].

There are several relevant and interesting studies of the integrable spin-s XXZ spin chains. The expression of eigenvalues of the spin-s XXX transfer matrix $t^{(2s,2s)}(\lambda)$ was derived by Babujian [17,18,21] through the algebraic Bethe-ansatz method. It was also derived by solving the series of functional relations among the spin-s transfer matrices [22]. The functional relations are systematically generalized to the T systems [49]. Recently, the algebraic Bethe-ansatz for the spin-s XXZ transfer matrix has been thoroughly reviewed and reconstructed from the viewpoint of the algebraic Bethe-ansatz of the $U(1)$-invariant integrable model [50,51]. Quite interestingly, it has also been applied to construct the invariant subspaces associated with the Ising-like spectra of the superintegrable chiral Potts model [52].

The content of the paper consists of the following. In Section 2, we introduce the R-matrix for the spin-1/2 XXZ spin chain. We then introduce conjugate basis vectors in order to formulate Hermitian elementary matrices $\widetilde{E}^{m,n}$ in the massless regime where $|q| = 1$. We define the massless higher-spin monodromy matrices $\widetilde{T}^{(\ell,2s)}(\lambda)$ in terms of the conjugate vectors, after reviewing the fusion construction of the massive higher-spin monodromy matrices $T^{(\ell,2s)}(\lambda)$ and higher-spin XXZ transfer matrices, $t^{(\ell,2s)}(\lambda)$, for $\ell = 1, 2, \ldots,$ as follows. We express the matrix elements of $T_{0,12\cdots N_s}^{(\ell,2s)}(\lambda)$ in terms of those of the spin-1/2 monodromy matrix $T_{0,12\cdots L}^{(1,1)}(\lambda)$. Here $T_{0,12\cdots L}^{(1,1)}(\lambda)$ is defined on the tensor product of the two-dimensional auxiliary space $V_0^{(1)}$ and the Lth tensor product of the 2-dimensional quantum space, $(V^{(1)})^{\otimes L}$. Here we recall $L = 2s \times N_s$. In the fusion construction [15], monodromy matrix $T_{0,12\cdots N_s}^{(1,2s)}(\lambda)$ acting on the N_s lattice sites is derived from monodromy matrix $T_{0,12\cdots L}^{(1,1)}(\lambda)$ acting on the $2sN_s$ lattice sites by setting inhomogeneous parameters w_j to N_s sets of complete $2s$-strings and by multiplying it by the N_sth tensor product of projection operators which project $(V^{(1)})^{\otimes 2s}$ to $V^{(2s)}$. In Section 3, we explain the method for calculating the expectation values of given products of spin-s local operators. We express the local operators in terms of global operators with inhomogeneous parameters $w_j^{(2s;\epsilon)}$, which are defined to be close to complete $2s$-strings with small deviations of $O(\epsilon)$, and evaluate the scalar products and the expectation values for the Bethe state with the same inhomogeneous parameters $w_j^{(2s;\epsilon)}$. Then, we obtain the expectation values, sending ϵ to 0. Here we note that the projection operators introduced in the fusion construction commute with the matrix elements of monodromy matrix $T^{(1,1)}$ of inhomogeneous parameters $w_j^{(2s;\epsilon)}$ with $O(\epsilon)$ corrections. In Section 4 we calculate the emptiness formation probability (EFP) for the spin-s XXZ spin chain for a large but finite chain, and then evaluate the matrix S which is introduced for expressing the EFP of an infinite spin-s XXZ chain in the massless regime with $\zeta < \pi/2s$. Here we solve explicitly the integral equations for the spin-s Gaudin matrix, expressing the $2s$-strings of the ground-state solution systematically in terms of the string centers. In Section 5, we present explicitly the multiple-integral representation of the spin-s EFP. We also derive it for the inhomogeneous chain where all the inhomogeneous parameters ξ_p are given by 0. For an illustration, we calculate the multiple-integral representation of spin-1 EFP for $m=1$, $\langle \widetilde{E}^{2,2}\rangle$, explicitly. In the XXX limit, the value of $\langle \widetilde{E}^{2,2}\rangle$ approaches 1/3, which is consistent with the XXX result of Ref. [13]. In Section 6, we present the multiple-integral representations of the integrable spin-s XXZ correlation functions. We express the correlation function of an arbitrary product of elementary matrices by a single term of multiple integrals. For instance, we calculate the multiple-integral representation of the spin-1 ground-state expectation value, $\langle \widetilde{E}^{1,1}\rangle$, explicitly, and show that it is consistent with the value of spin-1 EFP in Section 5, i.e. we show $\langle \widetilde{E}^{1,1}\rangle + 2\langle \widetilde{E}^{2,2}\rangle = 1$. Finally in Section 7, we give concluding remarks.

2. Fusion transfer matrices

2.1. R-matrix and the monodromy matrix of type $(1, 1^{\otimes L})$

Let us introduce the R-matrix of the XXZ spin chain [1,3–5]. We denote by $e^{a,b}$ a unit matrix that has only one nonzero element equal to 1 at entry (a,b) where $a,b=0,1$. Let V_1 and V_2 be two-dimensional vector spaces. The R-matrix acting on $V_1 \otimes V_2$ is given by

$$R^+(\lambda_1-\lambda_2)=\sum_{a,b,c,d=0,1} R^+(u)^{ab}_{cd} e^{a,c}\otimes e^{b,d} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(u) & c^-(u) & 0 \\ 0 & c^+(u) & b(u) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.1}$$

where $u=\lambda_1-\lambda_2$, $b(u)=\sinh u/\sinh(u+\eta)$ and $c^{\pm}(u)=\exp(\pm u)\sinh\eta/\sinh(u+\eta)$. In the massless regime, we set $\eta = i\zeta$ by a real number ζ, and we have $\Delta=\cos\zeta$. In the paper we mainly consider the region $0\leqslant\zeta<\pi/2s$. In the massive regime, we assign η a real nonzero number and we have $\Delta=\cosh\eta>1$. Here we remark that the $R^+(\lambda_1-\lambda_2)$ is compatible with the homogeneous grading of $U_q(\widehat{sl_2})$, which is explained in Appendix A [15].

We denote by $R^{(p)}(u)$ or simply by $R(u)$ the symmetric R-matrix where $c^{\pm}(u)$ of (2.1) are replaced by $c(u)=\sinh\eta/\sinh(u+\eta)$ [15]. The symmetric R-matrix is compatible with the affine quantum group $U_q(\widehat{sl_2})$ of the principal grading [15].

Let s be an integer or a half-integer. We shall mainly consider the tensor product $V_1^{(2s)}\otimes\cdots\otimes V_{N_s}^{(2s)}$ of $(2s+1)$-dimensional vector spaces $V_j^{(2s)}$ with $L=2sN_s$. In general, we consider the tensor product $V_0^{(2s_0)}\otimes V_1^{(2s_1)}\otimes\cdots\otimes V_r^{(2s_r)}$ with $2s_1+\cdots+2s_r=L$, where $V_j^{(2s_j)}$ have spectral parameters λ_j for $j = 1, 2, \ldots, r$. For a given set of matrix elements $A_{b,\beta}^{a,\alpha}$ for $a, b = 0, 1, \ldots, 2s_j$ and $\alpha, \beta = 0, 1, \ldots, 2s_k$, we define operator $A_{j,k}$ by

$$A_{j,k} = \sum_{a,b=1}^{\ell} \sum_{\alpha,\beta} A_{b,\beta}^{a,\alpha} I_0^{(2s_0)} \otimes I_1^{(2s_1)} \otimes \cdots \otimes I_{j-1}^{(2s_{j-1})} \\ \otimes e_j^{a,b} \otimes I_{j+1}^{(2s_{j+1})} \otimes \cdots \otimes I_{k-1}^{(2s_{k-1})} \otimes e_k^{\alpha,\beta} \otimes I_{k+1}^{(2s_{k+1})} \otimes \cdots \otimes I_r^{(2s_r)}. \tag{2.2}$$

We now consider the $(L+1)$th tensor product of spin-1/2 representations, which consists of the tensor product of auxiliary space $V_0^{(1)}$ and the Lth tensor product of quantum spaces $V_j^{(1)}$ for $j = 1, 2, \ldots, L$, i.e. $V_0^{(1)} \otimes (V_1^{(1)} \otimes \cdots \otimes V_L^{(1)})$. We call it the tensor product of type $(1, 1^{\otimes L})$ and denote it by the following symbol:

$$\left(1, 1^{\otimes L}\right) = (1, \overbrace{1, 1, \ldots, 1}^{L}). \tag{2.3}$$

Applying definition (2.2) for matrix elements $R(u)_{cd}^{ab}$ of a given R-matrix, we define R-matrices $R_{jk}(\lambda_j, \lambda_k) = R_{jk}(\lambda_j - \lambda_k)$ for integers j and k with $0 \leqslant j < k \leqslant L$. For integers j, k and ℓ with $0 \leqslant j < k < \ell \leqslant L$, the R-matrices satisfy the Yang–Baxter equations

$$R_{jk}(\lambda_j - \lambda_k) R_{j\ell}(\lambda_j - \lambda_\ell) R_{k\ell}(\lambda_k - \lambda_\ell) = R_{k\ell}(\lambda_k - \lambda_\ell) R_{j\ell}(\lambda_j - \lambda_\ell) R_{jk}(\lambda_j - \lambda_k). \tag{2.4}$$

We define the monodromy matrix of type $(1, 1^{\otimes L})$ associated with homogeneous grading by

$$T_{0,12\cdots L}^{(1,1+)}(\lambda_0; w_1, w_2, \ldots, w_L) = R_{0L}^{+}(\lambda_0 - w_L) \cdots R_{02}^{+}(\lambda_0 - w_2) R_{01}^{+}(\lambda_0 - w_1). \tag{2.5}$$

Here we have set $\lambda_j = w_j$ for $j = 1, 2, \ldots, L$, where w_j are arbitrary parameters. We call them inhomogeneous parameters. We have expressed the symbol of type $(1, 1^{\otimes L})$ as $(1, 1)$ in superscript. The symbol $(1, 1+)$ denotes that it is consistent with homogeneous grading. We express operator-valued matrix elements of the monodromy matrix as follows:

$$T_{0,12\cdots L}^{(1,1+)}\left(\lambda; \{w_j\}_L\right) = \begin{pmatrix} A_{12\cdots L}^{(1+)}(\lambda; \{w_j\}_L) & B_{12\cdots L}^{(1+)}(\lambda; \{w_j\}_L) \\ C_{12\cdots L}^{(1+)}(\lambda; \{w_j\}_L) & D_{12\cdots L}^{(1+)}(\lambda; \{w_j\}_L) \end{pmatrix}. \tag{2.6}$$

Here $\{w_j\}_L$ denotes the set of L parameters, $w_1, w_2, \ldots, w_L$. We also denote the matrix elements of the monodromy matrix by $[T_{0,12\cdots L}^{(1,1+)}(\lambda; \{w_j\}_L)]_{a,b}$ for $a, b = 0, 1$.

We derive the monodromy matrix consistent with principal grading, $T_{0,12\cdots L}^{(1,1\,p)}(\lambda; \{w_j\}_L)$, from that of homogeneous grading via similarity transformation $\chi_{01\cdots L}$ as follows [15]:

$$T_{0,12\cdots L}^{(1,1+)}\left(\lambda; \{w_j\}_L\right) = \chi_{0,12\cdots L} T_{0,12\cdots L}^{(1,1\,p)}\left(\lambda; \{w_j\}_L\right) \chi_{0,12\cdots L}^{-1} \\ = \begin{pmatrix} \chi_{12\cdots L} A_{12\cdots L}^{(1\,p)}(\lambda; \{w_j\}_L) \chi_{12\cdots L}^{-1} & e^{-\lambda_0} \chi_{12\cdots L} B_{12\cdots L}^{(1\,p)}(\lambda; \{w_j\}_L) \chi_{12\cdots L}^{-1} \\ e^{\lambda_0} \chi_{12\cdots L} C_{12\cdots L}^{(1\,p)}(\lambda; \{w_j\}_L) \chi_{12\cdots L}^{-1} & \chi_{12\cdots L} D_{12\cdots L}^{(1\,p)}(\lambda; \{w_j\}_L) \chi_{12\cdots L}^{-1} \end{pmatrix}. \tag{2.7}$$

Here $\chi_{01\cdots L} = \Phi_0 \Phi_1 \cdots \Phi_L$ and Φ_j are given by diagonal two-by-two matrices $\Phi_j = \mathrm{diag}(1, \exp(w_j))$ acting on $V_j^{(1)}$ for $j = 0, 1, \ldots, L$, and we set $w_0 = \lambda_0$. In Ref. [15] operator $A^{(1+)}(\lambda)$ has been written as $A^{+}(\lambda)$. Hereafter we shall often abbreviate the symbols p in superscripts which shows the principal grading, and denote $(2sp)$ simply by $(2s)$.

Let us introduce useful notation for expressing products of R-matrices as follows:

$$R^{(w)}_{1,23\cdots n} = R^{(w)}_{1n}\cdots R^{(w)}_{13}R^{(w)}_{12}, \qquad R^{(w)}_{12\cdots n-1,n} = R^{(w)}_{1n}R^{(w)}_{2n}\cdots R^{(w)}_{n-1\,n}. \tag{2.8}$$

Here $R^{(w)}_{ab}$ denote the R-matrix $R^{(w)}_{ab} = R^{(w)}_{ab}(\lambda_a - \lambda_b)$ for $a, b = 1, 2, \ldots, n$, where $w = +$ and $w = p$ in superscripts show the homogeneous and the principal grading, respectively. Then, the monodromy matrix of type $(1, 1^{\otimes L}\, w)$ is expressed as follows:

$$T^{(1,1\,w)}_{0,12\cdots L}\big(\lambda_0; \{w_j\}_L\big) = R^{(w)}_{0,12\cdots L}\big(\lambda_0; \{w_j\}_L\big) = R^{(w)}_{0L}R^{(w)}_{0L-1}\cdots R^{(w)}_{01}. \tag{2.9}$$

For instance we have $B^{(1\,w)}_{12\cdots L}(\lambda_0; \{w_j\}_L) = [R^{(1\,w)}_{0,12\cdots L}(\lambda_0; \{w_j\}_L)]_{0,1}$.

2.2. Projection operators and the massive fusion R-matrices

Let V_1 and V_2 be $(2s+1)$-dimensional vector spaces. We define permutation operator $\Pi_{1,2}$ by

$$\Pi_{1,2}v_1 \otimes v_2 = v_2 \otimes v_1, \quad v_1 \in V_1,\ v_2 \in V_2. \tag{2.10}$$

In the case of spin-$1/2$ representations, we define operator $\check{R}^+_{12}(\lambda_1 - \lambda_2)$ by

$$\check{R}^+_{12}(\lambda_1 - \lambda_2) = \Pi_{1,2}R^+_{12}(\lambda_1 - \lambda_2). \tag{2.11}$$

We now introduce projection operators $P^{(\ell)}_{12\cdots\ell}$ for $\ell \geqslant 2$. We define $P^{(2)}_{12}$ by $P^{(2)}_{12} = \check{R}^+_{1,2}(\eta)$. For $\ell > 2$ we define projection operators inductively with respect to ℓ as follows [46,23]:

$$P^{(\ell)}_{12\cdots\ell} = P^{(\ell-1)}_{12\cdots\ell-1}\check{R}^+_{\ell-1,\ell}\big((\ell-1)\eta\big)P^{(\ell-1)}_{12\cdots\ell-1}. \tag{2.12}$$

The projection operator $P^{(\ell)}_{12\cdots\ell}$ gives a q-analogue of the full symmetrizer of the Young operators for the Hecke algebra [46]. We shall show the idempotency: $(P^{(\ell)}_{12\cdots\ell})^2 = P^{(\ell)}_{12\cdots\ell}$ in Appendix B. Hereafter we denote $P^{(\ell)}_{12\cdots\ell}$ also by $P^{(\ell)}_1$ for short.

Applying projection operator $P^{(\ell)}_{a_1a_2\cdots a_\ell}$ to vectors in the tensor product $V^{(1)}_{a_1} \otimes V^{(1)}_{a_2} \otimes \cdots \otimes V^{(1)}_{a_\ell}$, we can construct the $(\ell+1)$-dimensional vector space $V^{(\ell)}_{a_1a_2\cdots a_\ell}$ associated with the spin-$\ell/2$ representation of $U_q(sl_2)$. For instance, we have $P^{(2)}_{a_1a_2}|+-\rangle_a = (q/[2]_q)\|2,1\rangle_a$, where we have introduced $|+-\rangle_a = |0\rangle_{a_1} \otimes |1\rangle_{a_2}$. The symbols such as q-integers are defined in Appendix C. Moreover, the basis vectors $\|\ell, n\rangle$ $(n = 0, 1, \ldots, \ell)$ and their dual vectors $\langle \ell, n\|$ are given for arbitrary nonzero integers ℓ in Appendix C. We denote $V^{(\ell)}_{a_1a_2\cdots a_\ell}$ also by $V^{(\ell)}_a$ or $V^{(\ell)}_0$ for short.

Since $P^{(\ell)}_{12\cdots\ell}$ is consistent with the spin-$\ell/2$ representation of $U_q(sl(2))$ (see (C.6)), we have

$$P^{(\ell)}_{12\cdots\ell} = \sum_{n=0}^{\ell} \|\ell, n\rangle\langle \ell, n\|. \tag{2.13}$$

Applying projection operator $P^{(2s)}_{2s(j-1)+1\cdots 2s(j-1)+2s}$ to tensor product $V^{(1)}_{2s(j-1)+1} \otimes \cdots \otimes V^{(1)}_{2s(j-1)+2s}$, we construct the spin-$s$ representation $V^{(2s)}_{2s(j-1)+1\cdots 2s(j-1)+2s}$. We denote it also by $V^{(2s)}_j$, briefly.

In the tensor product of quantum spaces $V_1^{(2s)} \otimes \cdots \otimes V_{N_s}^{(2s)}$, we define $P_{12\cdots L}^{(2s)}$ by

$$P_{12\cdots L}^{(2s)} = \prod_{i=1}^{N_s} P_{2s(i-1)+1}^{(2s)}. \tag{2.14}$$

Here we recall $L = 2sN_s$. We have put $2s$ in place of ℓ.

We now introduce the massive fusion R-matrix $R_{0,j}^{(\ell,2s+)}$ on the tensor product $V_0^{(\ell)} \otimes V_j^{(2s)}$ ($j = 1, 2, \ldots, N_s$). It is valid in the massive regime with $\Delta > 1$. We first set rapidities λ_{a_j} of auxiliary spaces $V_{a_j}^{(1)}$ by $\lambda_{a_k} = \lambda_{a_1} - (k-1)\eta$ for $k = 1, 2, \ldots, \ell - 1$, and then rapidities $\lambda_{2s(j-1)+k}$ of quantum spaces $V_{2s(j-1)+k}^{(1)}$ by $\lambda_{2s(j-1)+k} = \lambda_{2s(j-1)+1} - (k-1)\eta$ for $k = 1, 2, \ldots, 2s$ and $j = 1, 2, \ldots, N_s$. We define the massive fusion R-matrix $R_{0,j}^{(\ell,2s+)}$ as follows:

$$\begin{aligned} &R_{0j}^{(\ell,2s+)}(\lambda_{a_1} - \lambda_{2s(j-1)+1}) \\ &\quad = P_{a_1\cdots a_\ell}^{(\ell)} P_{2s(j-1)+1}^{(2s)} R_{a_1\cdots a_\ell, 2s(j-1)+1\cdots 2sj}^{+} P_{a_1\cdots a_\ell}^{(\ell)} P_{2s(j-1)+1}^{(2s)} \\ &\quad = P_{a_1\cdots a_\ell}^{(\ell)} P_{2s(j-1)+1}^{(2s)} R_{a_1\cdots a_\ell, 2sj}^{+} \cdots R_{a_1\cdots a_\ell, 2s(j-1)+2}^{+} R_{a_1\cdots a_\ell, 2s(j-1)+1}^{+} P_{a_1\cdots a_\ell}^{(\ell)} P_{2s(j-1)+1}^{(2s)}. \end{aligned} \tag{2.15}$$

2.3. Conjugate vectors and the massless fusion R-matrices

In order to construct Hermitian elementary matrices in the massless regime where $|q| = 1$, we now introduce vectors $\widetilde{\|\ell, n\rangle}$ which are Hermitian conjugate to $\langle \ell, n\|$ when $|q| = 1$ for positive integers ℓ with $n = 0, 1, \ldots, \ell$. Setting the norm of $\widetilde{\|\ell, n\rangle}$ such that $\langle \ell, n\| \widetilde{\|\ell, n\rangle} = 1$, we have

$$\begin{aligned} &\widetilde{\|\ell, n\rangle} \\ &\quad = \sum_{1 \leqslant i_1 < \cdots < i_n \leqslant \ell} \sigma_{i_1}^{-} \cdots \sigma_{i_n}^{-} |0\rangle q^{-(i_1 + \cdots + i_n) + n\ell - n(n-1)/2} \begin{bmatrix} \ell \\ n \end{bmatrix}_q q^{-n(\ell-n)} \begin{pmatrix} \ell \\ n \end{pmatrix}^{-1}. \end{aligned} \tag{2.16}$$

Here we have denoted the binomial coefficients for integers ℓ and n with $0 \leqslant n \leqslant \ell$ as follows:

$$\begin{pmatrix} \ell \\ n \end{pmatrix} = \frac{\ell!}{(\ell - n)!n!}. \tag{2.17}$$

The q-binomial coefficients are defined in Appendix C. Dual vectors $\widetilde{\langle \ell, n\|}$, which are conjugate to $\|\ell, n\rangle$, are defined in Appendix C, and we have

$$\widetilde{\langle \ell, n\|} \, \widetilde{\|\ell, n\rangle} = \begin{bmatrix} \ell \\ n \end{bmatrix}_q^2 \begin{pmatrix} \ell \\ n \end{pmatrix}^{-2}. \tag{2.18}$$

They are determined by the action of $X^{\pm}$ with opposite coproduct: $\Delta^{op} = \tau \circ \Delta$. For instance, we have $\widetilde{\|\ell, n\rangle} = \mathrm{const}\, \Delta^{op}(X^-)^n \|\ell, 0\rangle$. Here $X^{\pm}$ and Δ^{op} are defined in Appendix A.

For an illustration, in the spin-1 case, the basis vectors $\|2, n\rangle$ ($n = 0, 1, 2$) are given by [15]

$$\|2, 0\rangle = |++\rangle, \qquad \|2, 1\rangle = |+-\rangle + q^{-1}|-+\rangle, \qquad \|2, 2\rangle = |--\rangle. \tag{2.19}$$

Here $|+-\rangle$ denotes $|0\rangle_1 \otimes |1\rangle_2$, briefly. The conjugate vectors $\widetilde{\|2, n\rangle}$ ($n = 0, 1, 2$) are given by

$$\widetilde{\|2,0\rangle} = |++\rangle, \qquad \widetilde{\|2,1\rangle} = \big(|+-\rangle + q|-+\rangle\big)\frac{[2]_q}{2q}, \qquad \widetilde{\|2,2\rangle} = |--\rangle. \tag{2.20}$$

In the massless regime, operator $\widetilde{\|2,1\rangle}\langle 2,1\|$ is Hermitian while $\|2,1\rangle\langle 2,1\|$ is not.

Let us now introduce another set of projection operators $\widetilde{P}^{(\ell)}_{1\cdots\ell}$ as follows:

$$\widetilde{P}^{(\ell)}_{1\cdots\ell} = \sum_{n=0}^{\ell} \widetilde{\|\ell,n\rangle}\langle \ell,n\|. \tag{2.21}$$

Projector $\widetilde{P}^{(\ell)}_{1\cdots\ell}$ is idempotent: $(\widetilde{P}^{(\ell)}_{1\cdots\ell})^2 = \widetilde{P}^{(\ell)}_{1\cdots\ell}$. In the massless regime where $|q|=1$, it is Hermitian: $(\widetilde{P}^{(\ell)}_{1\cdots\ell})^\dagger = \widetilde{P}^{(\ell)}_{1\cdots\ell}$. From (2.13) and (2.21), we show the following properties:

$$P^{(\ell)}_{12\cdots\ell}\widetilde{P}^{(\ell)}_{1\cdots\ell} = P^{(\ell)}_{12\cdots\ell}, \tag{2.22}$$
$$\widetilde{P}^{(\ell)}_{1\cdots\ell}P^{(\ell)}_{12\cdots\ell} = \widetilde{P}^{(\ell)}_{1\cdots\ell}. \tag{2.23}$$

In the tensor product of quantum spaces, $V_1^{(2s)} \otimes \cdots \otimes V_{N_s}^{(2s)}$, we define $\widetilde{P}^{(2s)}_{12\cdots L}$ by

$$\widetilde{P}^{(2s)}_{12\cdots L} = \prod_{i=1}^{N_s} \widetilde{P}^{(2s)}_{2s(i-1)+1}. \tag{2.24}$$

Here we recall $L = 2sN_s$ such as for (2.14).

We define the massless fusion R-matrix $\widetilde{R}^{(\ell,2s+)}_{0,j}$, applying projection operators $\widetilde{P}$ consisting of conjugate vectors to the product of R-matrices, as follows:

$$\begin{aligned}
&\widetilde{R}^{(\ell,2s+)}_{0\,j}(\lambda_{a_1} - w_{2s(j-1)+1}) \\
&\quad = \widetilde{P}^{(\ell)}_{a_1\cdots a_\ell}\widetilde{P}^{(2s)}_{2s(j-1)+1}R^{+}_{a_1\cdots a_\ell,2s(j-1)+1\cdots 2sj}\widetilde{P}^{(\ell)}_{a_1\cdots a_\ell}\widetilde{P}^{(2s)}_{2s(j-1)+1} \\
&\quad = \widetilde{P}^{(\ell)}_{a_1\cdots a_\ell}\widetilde{P}^{(2s)}_{2s(j-1)+1}R^{+}_{a_1\cdots a_\ell,2sj}\cdots R^{+}_{a_1\cdots a_\ell,2s(j-1)+2}R^{+}_{a_1\cdots a_\ell,2s(j-1)+1}\widetilde{P}^{(\ell)}_{a_1\cdots a_\ell}\widetilde{P}^{(2s)}_{2s(j-1)+1}.
\end{aligned} \tag{2.25}$$

We should remark that the massless fusion R-matrix $\widetilde{R}^{(\ell,2s)}$ and the massive fusion R-matrix $R^{(\ell,2s)}$ have the same matrix elements. Some examples are shown in Appendix D.

2.4. Higher-spin monodromy matrix of type $(\ell, (2s)^{\otimes N_s})$

We now set the inhomogeneous parameters w_j for $j=1,2,\ldots,L$, as N_s sets of complete $2s$-strings [15]. We define $w^{(2s)}_{(b-1)\ell+\beta}$ for $\beta = 1,\ldots,2s$, as follows:

$$w^{(2s)}_{2s(b-1)+\beta} = \xi_b - (\beta-1)\eta, \quad \text{for } b = 1,2,\ldots,N_s. \tag{2.26}$$

We shall define the monodromy matrix of type $(1,(2s)^{\otimes N_s})$ associated with homogeneous grading. We first define the massless monodromy matrix by

$$\begin{aligned}
\widetilde{T}^{(1,2s+)}_{0,12\cdots N_s}\big(\lambda_0;\{\xi_b\}_{N_s}\big) &= \widetilde{P}^{(2s)}_{12\cdots L}R^{(1,1+)}_{0,1\ldots L}\big(\lambda_0;\{w^{(2s)}_j\}_L\big)\widetilde{P}^{(2s)}_{12\cdots L} \\
&= \begin{pmatrix} \widetilde{A}^{(2s+)}(\lambda;\{\xi_b\}_{N_s}) & \widetilde{B}^{(2s+)}(\lambda;\{\xi_b\}_{N_s}) \\ \widetilde{C}^{(2s+)}(\lambda;\{\xi_b\}_{N_s}) & \widetilde{D}^{(2s+)}(\lambda;\{\xi_b\}_{N_s}) \end{pmatrix}.
\end{aligned} \tag{2.27}$$

Here, the (0, 0) element is given by $\widetilde{A}^{(2s+)}(\lambda; \{\xi_b\}_{N_s}) = \widetilde{P}^{(2s)}_{12\cdots L} A^{(1+)}(\lambda; \{w_j^{(2s)}\}_L) \widetilde{P}^{(2s)}_{12\cdots L}$. We then define the massive monodromy matrix by

$$T^{(1,2s+)}_{0,12\cdots N_s}\big(\lambda_0; \{\xi_b\}_{N_s}\big) = P^{(2s)}_{12\cdots L} R^{(1,1+)}_{0,1\ldots L}\big(\lambda_0; \big\{w_j^{(2s)}\big\}_L\big) P^{(2s)}_{12\cdots L}$$
$$= \begin{pmatrix} A^{(2s+)}(\lambda; \{\xi_b\}_{N_s}) & B^{(2s+)}(\lambda; \{\xi_b\}_{N_s}) \\ C^{(2s+)}(\lambda; \{\xi_b\}_{N_s}) & D^{(2s+)}(\lambda; \{\xi_b\}_{N_s}) \end{pmatrix}. \tag{2.28}$$

Let us introduce a set of $2s$-strings with small deviations from the set of complete $2s$-strings

$$w^{(2s;\,\epsilon)}_{2s(b-1)+\beta} = \xi_b - (\beta - 1)\eta + \epsilon r_b^{(\beta)}, \quad \text{for } b = 1, 2, \ldots, N_s, \text{ and } \beta = 1, 2, \ldots, 2s. \tag{2.29}$$

Here ϵ is very small and $r_b^{(\beta)}$ are generic parameters. We express the elements of the monodromy matrix $T^{(1,1)}$ with inhomogeneous parameters given by $w_j^{(2s;\epsilon)}$ for $j = 1, 2, \ldots, L$ as follows:

$$T^{(1,1+)}_{0,12\cdots L}\big(\lambda; \big\{w_j^{(2s;\epsilon)}\big\}_L\big) = \begin{pmatrix} A^{(2s+;\epsilon)}_{12\cdots L}(\lambda) & B^{(2s+;\epsilon)}_{12\cdots L}(\lambda) \\ C^{(2s+;\epsilon)}_{12\cdots L}(\lambda) & D^{(2s+;\epsilon)}_{12\cdots L}(\lambda) \end{pmatrix}. \tag{2.30}$$

Here we recall that $A^{(2s+;\epsilon)}_{12\cdots L}(\lambda)$ denotes $A^{(1+)}_{12\cdots L}(\lambda; \{w_j^{(2s;\epsilon)}\}_L)$. We also remark the following:

$$\widetilde{A}^{(2s+)}_{12\cdots N_s}\big(\lambda; \{\xi_p\}_{N_s}\big) = \lim_{\epsilon\to 0} \widetilde{P}^{(2s)}_{12\cdots L} A^{(2s+;\epsilon)}_{12\cdots L}\big(\lambda; \big\{w_j^{(2s;\epsilon)}\big\}_L\big) \widetilde{P}^{(2s)}_{12\cdots L}. \tag{2.31}$$

Let us express the tensor product $V_0^{(\ell)} \otimes (V_1^{(2s)} \otimes \cdots \otimes V_{N_s}^{(2s)})$, by the following symbol

$$\big(\ell, (2s)^{\otimes N_s}\big) = (\ell, \overbrace{2s, 2s, \ldots, 2s}^{N_s}). \tag{2.32}$$

Here we recall that $V_0^{(\ell)}$ abbreviates $V^{(\ell)}_{a_1a_2\cdots a_\ell}$. In the case of auxiliary space $V_0^{(\ell)}$ we define the massless monodromy matrix of type $(\ell, (2s)^{\otimes N_s})$ by

$$\widetilde{T}^{(\ell,2s+)}_{0,12\cdots N_s}$$
$$= \widetilde{P}^{(\ell)}_{a_1a_2\cdots a_\ell} \widetilde{T}^{(1,2s+)}_{a_1,12\cdots N_s}(\lambda_{a_1}) \widetilde{T}^{(1,2s+)}_{a_2,12\cdots N_s}(\lambda_{a_1} - \eta) \cdots \widetilde{T}^{(1,2s+)}_{a_\ell,12\cdots N_s}\big(\lambda_{a_1} - (\ell - 1)\eta\big) \widetilde{P^{(\ell)}}_{a_1a_2\cdots a_\ell}, \tag{2.33}$$

and the massive monodromy matrix of type $(\ell, (2s)^{\otimes N_s})$ by

$$T^{(\ell,2s+)}_{0,12\cdots N_s}$$
$$= P^{(\ell)}_{a_1a_2\cdots a_\ell} T^{(1,2s+)}_{a_1,12\cdots N_s}(\lambda_{a_1}) T^{(1,2s+)}_{a_2,12\cdots N_s}(\lambda_{a_1} - \eta) \cdots T^{(1,2s+)}_{a_\ell,12\cdots N_s}\big(\lambda_{a_1} - (\ell - 1)\eta\big) P^{(\ell)}_{a_1a_2\cdots a_\ell}. \tag{2.34}$$

For instance, the (0, 1) element of the massive monodromy matrix $T^{(2,2s+)}(\lambda)$ is given by

$$\langle 2, 0\| T^{(2,2s+)}_{a_1a_2,12\cdots N_s}(\lambda) \|2, 1\rangle$$
$$= A^{(2s+)}_{a_1}(\lambda) B^{(2s+)}_{a_2}(\lambda - \eta) + q^{-1} B^{(2s+)}_{a_1}(\lambda) A^{(2s+)}_{a_2}(\lambda - \eta). \tag{2.35}$$

2.5. Series of commuting higher-spin transfer matrices

Suppose that $|\ell, m\rangle$ for $m = 0, 1, \ldots, \ell$, are the orthonormal basis vectors of $V^{(\ell)}$, and their dual vectors are given by $\langle \ell, m|$ for $m = 0, 1, \ldots, \ell$. We define the trace of operator A over the space $V^{(\ell)}$ by

$$\mathrm{tr}_{V^{(\ell)}} A = \sum_{m=0}^{\ell} \langle \ell, m|A|\ell, m\rangle. \tag{2.36}$$

The trace of A over $V^{(\ell)}$ is equivalent to the trace of A over the ℓth tensor product of $V^{(1)}$, $(V^{(1)})^{\otimes \ell}$, multiplied by a projector $P^{(\ell)}$ (or $\widetilde{P}^{(\ell)}$) as follows:

$$\mathrm{tr}_{V^{(\ell)}} A = \mathrm{tr}_{(V^{(1)})^{\otimes \ell}}\big(P^{(\ell)} A\big) = \sum_{a_1, \ldots, a_\ell = 0, 1} \big(P^{(\ell)} A\big)_{a_1 \cdots a_\ell}^{a_1 \cdots a_\ell}. \tag{2.37}$$

It follows from (2.13) that the trace with respect to $V^{(\ell)}$ is given by (2.36).

We define the massive transfer matrix of type $(\ell, (2s)^{\otimes N_s})$ by

$$\begin{aligned} t_{12\cdots N_s}^{(\ell, 2s+)}(\lambda) &= \mathrm{tr}_{V^{(\ell)}}\big(T_{0,12\cdots N_s}^{(\ell, 2s+)}(\lambda)\big) \\ &= \sum_{n=0}^{\ell} {}_a\langle \ell, n\| T_{a_1, 12\cdots N_s}^{(1,2s+)}(\lambda) T_{a_2, 12\cdots N_s}^{(1,2s+)}(\lambda - \eta) \cdots T_{a_\ell, 12\cdots N_s}^{(1,2s+)}\big(\lambda - (\ell - 1)\eta\big) \|\ell, n\rangle_a, \end{aligned} \tag{2.38}$$

and the massless transfer matrix of type $(\ell, (2s)^{\otimes N_s})$ by

$$\begin{aligned} \tilde{t}_{12\cdots N_s}^{(\ell, 2s+)}(\lambda) &= \mathrm{tr}_{V^{(\ell)}}\big(\widetilde{T}_{0,12\cdots N_s}^{(\ell, 2s+)}(\lambda)\big) \\ &= \sum_{n=0}^{\ell} {}^a\langle \ell, n\| \widetilde{T}_{a_1, 12\cdots N_s}^{(1,2s+)}(\lambda) \widetilde{T}_{a_2, 12\cdots N_s}^{(1,2s+)}(\lambda - \eta) \cdots \widetilde{T}_{a_\ell, 12\cdots N_s}^{(1,2s+)}\big(\lambda - (\ell - 1)\eta\big) \widetilde{\|\ell, n\rangle}_a. \end{aligned} \tag{2.39}$$

It follows from the Yang–Baxter equations that the higher-spin transfer matrices commute in the tensor product space $V_1^{(2s)} \otimes \cdots \otimes V_{N_s}^{(2s)}$, which is derived by applying projection operator $P_{12\cdots L}^{(2s)}$ to $V_1^{(1)} \otimes \cdots \otimes V_L^{(1)}$. For instance, for the massless transfer matrices, making use of (2.22) and (2.23) we show

$$P_{12\cdots L}^{(2s)}\big[\tilde{t}_{12\cdots N_s}^{(\ell, 2s+)}(\lambda), \tilde{t}_{12\cdots N_s}^{(m, 2s+)}(\mu)\big] = 0, \quad \text{for } \ell, m \in \mathbf{Z}_{\geqslant 0}. \tag{2.40}$$

Therefore, for the massless transfer matrices, the eigenvectors of $\tilde{t}_{12\cdots N_s}^{(1,2s+)}(\lambda)$ constructed by applying $\widetilde{B}^{(2s+)}(\lambda)$ to the vacuum $|0\rangle$ also diagonalize the higher-spin transfer matrices, in particular, $\tilde{t}_{12\cdots N_s}^{(2s,2s+)}(\lambda)$. Thus, we construct the ground state of the higher-spin Hamiltonian in terms of operators $\widetilde{B}^{(2s+)}(\lambda)$, which are the (0, 1) element of the monodromy matrix $\widetilde{T}^{(1,2s+)}$.

2.6. The integrable higher-spin Hamiltonians

We now discuss the integrable massless spin-s XXZ Hamiltonian. For $(2s+1)$-dimensional vector spaces $V_1^{(2s)}$ and $V_2^{(2s)}$, we can show that the massive spin-s fusion R-matrix $R_{12}^{(2s,2s+)}(u)$ at $u = 0$ becomes the permutation operator $\Pi_{1,2}$ for $V_1^{(2s)} \otimes V_2^{(2s)}$. Furthermore, operator $\check{R}_{1,2}^{(2s,2s+)}(u) = \Pi_{1,2} R_{12}^{(2s,2s+)}(u)$ has the following spectral decomposition:

$$\check{R}_{1,2}^{(2s,2s+)}(u)=\sum_{j=0}^{2s}\rho_{4s-2j}(u)\big(P_{4s-2j}^{2s,2s}\big)_{1,2}, \tag{2.41}$$

where operator $(P_{4s-2j}^{2s,2s})_{1,2}$ projects $V_1^{(2s)}\otimes V_2^{(2s)}$ to spin-$(2s-j)$ representation for $j=0,1,\dots,2s$. Functions $\rho_{4s-2j}(u)$ are given by [45]

$$\rho_{4s-2j}(u)=\prod_{k=2s-j+1}^{2s}\frac{\sinh(k\eta-u)}{\sinh(k\eta+u)}. \tag{2.42}$$

The massless spin-s R-matrix is thus given by

$$\tilde{\check{R}}_{i,i+1}^{(2s,2s+)}(u)=\sum_{j=0}^{2s}\rho_{4s-2j}(u)\widetilde{P}_{2s(i-1)+1}^{(2s)}\widetilde{P}_{2si+1}^{(2s)}\cdot\big(P_{4s-2j}^{2s,2s}\big)_{i,i+1}. \tag{2.43}$$

It is easy to show that the massless spin-s R-matrix $\widetilde{R}_{12}^{(2s,2s+)}(u)$ becomes the permutation operator at $u=0$: $\widetilde{R}_{12}^{(2s,2s+)}(0)=\Pi_{1,2}$. Therefore, putting inhomogeneous parameters $\xi_p=0$ for $p=1,2,\dots,N_s$, we show that the transfer matrix $\tilde{t}_{12\cdots N_s}^{(2s,2s+)}(\lambda)$ becomes the shift operator at $\lambda=0$. We thus derive the massless spin-s XXZ Hamiltonian by the logarithmic derivative of the massless spin-s transfer matrix, similarly as for the massive case

$$\begin{aligned}\mathcal{H}_{\mathrm{XXZ}}^{(2s)}&=\frac{d}{d\lambda}\log\tilde{t}_{12\cdots N_s}^{(2s,2s+)}(\lambda)\bigg|_{\lambda=0,\xi_j=0}=\sum_{i=1}^{N_s}\frac{d}{du}\tilde{\check{R}}_{i,i+1}^{(2s,2s)}(u)\bigg|_{u=0}\\&=\sum_{i=1}^{N_s}\sum_{j=0}^{2s}\frac{d\rho_{4s-2j}}{du}(0)\widetilde{P}_{2s(i-1)+1}^{(2s)}\widetilde{P}_{2si+1}^{(2s)}\cdot\big(P_{4s-2j}^{2s,2s}\big)_{i,i+1}.\end{aligned} \tag{2.44}$$

3. Higher-spin expectation values

3.1. Algebraic Bethe-ansatz

In terms of the vacuum vector $|0\rangle$ where all spins are up, we define functions $a(\lambda)$ and $d(\lambda)$ by

$$\begin{aligned}A^{(1\,p)}\big(\lambda;\{w_j\}_L\big)|0\rangle&=a\big(\lambda;\{w_j\}_L\big)|0\rangle,\\D^{(1\,p)}\big(\lambda;\{w_j\}_L\big)|0\rangle&=d\big(\lambda;\{w_j\}_L\big)|0\rangle.\end{aligned} \tag{3.1}$$

We have $a(\lambda;\{w_j\}_L)=1$ and

$$d\big(\lambda;\{w_j\}_L\big)=\prod_{j=1}^{L}b(\lambda,w_j). \tag{3.2}$$

Here $b(\lambda,\mu)=b(\lambda-\mu)$. For the homogeneous grading ($w=+$) and the principal grading ($w=p$), it is easy to show the following relations:

$$\begin{aligned}A^{(2s\,w)}(\lambda)|0\rangle&=\widetilde{A}^{(2s\,w)}(\lambda)|0\rangle=a^{(2s)}\big(\lambda;\{\xi_k\}\big)|0\rangle,\\D^{(2s\,w)}(\lambda)|0\rangle&=\widetilde{D}^{(2s\,w)}(\lambda)|0\rangle=d^{(2s)}\big(\lambda;\{\xi_k\}\big)|0\rangle,\end{aligned} \tag{3.3}$$

where $a^{(2s)}(\lambda;\{\xi_k\})$ and $d^{(2s)}(\lambda;\{\xi_k\})$ are given by

$$a^{(2s)}\big(\lambda;\{\xi_k\}\big)=a\big(\lambda;\big\{w_j^{(2s)}\big\}\big)=1,$$
$$d^{(2s)}\big(\lambda;\{\xi_k\}\big)=d\big(\lambda;\big\{w_j^{(2s)}\big\}\big)=\prod_{p=1}^{N_s} b_{2s}(\lambda,\xi_p). \tag{3.4}$$

Here we have defined $b_t(\lambda,\mu)$ by $b_t(\lambda,\mu)=\sinh(\lambda-\mu)/\sinh(\lambda-\mu+t\eta)$. Here we recall $b(u)=b_1(u)=\sinh u/\sinh(u+\eta)$.

In the massless regime, we define the Bethe vectors $|\{\widetilde{\lambda_\alpha}\}_M^{(2s\,w)}\rangle$ for $w=+$ and p, and their dual vectors $\langle\{\widetilde{\lambda_\alpha}\}_M^{(2s\,w)}|$ for $w=+$ and p, as follows:

$$\big|\{\widetilde{\lambda_\alpha}\}_M^{(2s\,w)}\big\rangle=\prod_{\alpha=1}^{M}\widetilde{B}^{(2s\,w)}(\lambda_\alpha)|0\rangle, \tag{3.5}$$

$$\big\langle\{\widetilde{\lambda_\alpha}\}_M^{(2s\,w)}\big|=\langle 0|\prod_{\alpha=1}^{M}\widetilde{C}^{(2s\,w)}(\lambda_\alpha). \tag{3.6}$$

Here we recall $\widetilde{B}^{(2s+)}(\lambda_\alpha)=\widetilde{P}_{1\cdots L}^{(2s)}B^{(1+)}(\lambda_\alpha,\{w_k\}_L)\widetilde{P}_{1\cdots L}^{(2s)}$. The Bethe vector (3.5) gives an eigenvector of the massless transfer matrix

$$\tilde{t}^{(1,2s\,w)}\big(\mu;\{\xi_p\}_{N_s}\big)=\widetilde{A}^{(2s\,w)}\big(\mu;\{\xi_p\}_{N_s}\big)+\widetilde{D}^{(2s\,w)}\big(\mu;\{\xi_p\}_{N_s}\big) \tag{3.7}$$

for $w=+$ and $w=p$ with the following eigenvalue:

$$\Lambda^{(1,2s\,w)}(\mu)=\prod_{j=1}^{M}\frac{\sinh(\lambda_j-\mu+\eta)}{\sinh(\lambda_j-\mu)}+\prod_{p=1}^{N_s}b_{2s}(\mu,\xi_p)\cdot\prod_{j=1}^{M}\frac{\sinh(\mu-\lambda_j+\eta)}{\sinh(\mu-\lambda_j)}, \tag{3.8}$$

if rapidities $\{\lambda_j\}_M$ satisfy the Bethe-ansatz equations

$$\prod_{p=1}^{N_s}b_{2s}^{-1}(\lambda_j,\xi_p)=\prod_{k\neq j}\frac{b(\lambda_k,\lambda_j)}{b(\lambda_j,\lambda_k)}\quad (j=1,\dots,M). \tag{3.9}$$

Let us denote by $|\{\lambda_\alpha(\epsilon)\}_M^{(2s\,w;\epsilon)}\rangle$ the Bethe vector of M Bethe roots $\{\lambda_j(\epsilon)\}_M$ for $w=+,\,p$:

$$\big|\big\{\lambda_\alpha(\epsilon)\big\}_M^{(2s\,w;\epsilon)}\big\rangle=B^{(2s\,w;\epsilon)}\big(\lambda_1(\epsilon)\big)\cdots B^{(2s\,w;\epsilon)}\big(\lambda_M(\epsilon)\big)|0\rangle, \tag{3.10}$$

where rapidities $\{\lambda_j(\epsilon)\}_M$ satisfy the Bethe-ansatz equations with inhomogeneous parameters $w_j^{(2s;\epsilon)}$ as follows:

$$\frac{a(\lambda_j(\epsilon);\{w_k^{(2s;\epsilon)}\}_L)}{d(\lambda_j(\epsilon);\{w_k^{(2s;\epsilon)}\}_L)}=\prod_{k=1;\,k\neq j}^{M}\frac{b(\lambda_k(\epsilon),\lambda_j(\epsilon))}{b(\lambda_j(\epsilon),\lambda_k(\epsilon))}. \tag{3.11}$$

It gives an eigenvector of the transfer matrix

$$t^{(1,1\,w)}\big(\mu;\big\{w_j^{(2s;\epsilon)}\big\}_L\big)=A^{(2s\,w;\epsilon)}\big(\mu;\big\{w_j^{(2s;\epsilon)}\big\}_L\big)+D^{(2s\,w;\epsilon)}\big(\mu;\big\{w_j^{(2s;\epsilon)}\big\}_L\big) \tag{3.12}$$

with the following eigenvalue:

$$\Lambda^{(1,1\,w)}\big(\mu;\,\big\{w_j^{(2s;\epsilon)}\big\}_L\big) = \prod_{j=1}^{M} \frac{\sinh(\lambda_j(\epsilon)-\mu+\eta)}{\sinh(\lambda_j(\epsilon)-\mu)} + \prod_{j=1}^{L} b\big(\mu, w_j^{(2s;\epsilon)}\big) \cdot \prod_{j=1}^{M} \frac{\sinh(\mu-\lambda_j(\epsilon)+\eta)}{\sinh(\mu-\lambda_j(\epsilon))}. \tag{3.13}$$

Let us assume that in the limit of ϵ going to 0, the set of Bethe roots $\{\lambda_j(\epsilon)\}_M$ is given by $\{\lambda_j\}_M$. Then, we have

$$P_{12\cdots L}^{(2s)}\big|\widetilde{\{\lambda_j\}}_M^{(2s+)}\big\rangle = \lim_{\epsilon\to 0} P_{12\cdots L}^{(2s)}\big|\big\{\lambda_j(\epsilon)\big\}_M^{(2s+;\epsilon)}\big\rangle. \tag{3.14}$$

3.2. Hermitian elementary matrices $\widetilde{E}_i^{m,n(2s+)}$ in the massless regime

We define massless elementary matrices $\widetilde{E}^{m,n(2s+)}$ for $m, n = 0, 1, \ldots, 2s$, in the spin-s representation of $U_q(sl_2)$ as follows:

$$\widetilde{E}^{m,n(2s+)} = \widetilde{\|\ell, m\rangle}\langle \ell, n\|. \tag{3.15}$$

In the tensor product space, $(V^{(2s)})^{\otimes N_s}$, we define $\widetilde{E}_i^{m,n(2s+)}$ for $i = 1, 2, \ldots, N_s$ by

$$\widetilde{E}_i^{m,n(2s+)} = \big(I^{(2s)}\big)^{\otimes(i-1)} \otimes \widetilde{E}^{m,n(2s+)} \otimes \big(I^{(2s)}\big)^{\otimes(N_s-i)}. \tag{3.16}$$

Elementary matrices $\widetilde{E}^{n,n(2s+)}$ for $n = 0, 1, \ldots, 2s$, are Hermitian in the massless regime. In fact, when $|q| = 1$, for $m, n = 0, 1, \ldots, 2s$, we have

$$\big(\widetilde{E}^{m,n(2s+)}\big)^{\dagger} = \begin{bmatrix} 2s \\ m \end{bmatrix}_q^2 \begin{bmatrix} 2s \\ n \end{bmatrix}_q^{-2} \binom{2s}{m}^{-1} \binom{2s}{n} \widetilde{E}^{n,m(2s+)}. \tag{3.17}$$

We can express any given spin-s local operator of the massless case in terms of the spin-1/2 global operators by a method similar to the massive case [15]. For $m = n$, we have

$$\widetilde{E}_i^{n,n(2s+)} = \binom{2s}{n} \widetilde{P}_{1\cdots L}^{(2s)} \prod_{\alpha=1}^{(i-1)2s} \big(A^{(1+)} + D^{(1+)}\big)(w_\alpha) \prod_{k=1}^{n} D^{(1+)}(w_{(i-1)2s+k}) \times \prod_{k=n+1}^{2s} A^{(1+)}(w_{(i-1)2s+k}) \prod_{\alpha=i2s+1}^{2sN_s} \big(A^{(1+)} + D^{(1+)}\big)(w_\alpha) \widetilde{P}_{1\cdots L}^{(2s)}. \tag{3.18}$$

Formulas expressing $\widetilde{E}^{m,n(2s+)}$ for $m > n$ or $m < n$ are given in Appendix E.

When we evaluate expectation values, we want to remove the projection operators introduced in order to express the spin-s local operator in terms of spin-1/2 global operators such as in (3.18). Then, we shall make use of the following lemma.

Lemma 3.1. *Projection operators $P_{12\cdots L}^{(2s)}$ and $\widetilde{P}_{12\cdots L}^{(2s)}$ commute with the matrix elements of the monodromy matrix $T_{0,12\cdots L}^{(1,1+)}(\lambda; \{w_j^{(2s;\epsilon)}\}_L)$ such as $A^{(2s+;\epsilon)}(\lambda)$ in the limit of ϵ going to* 0

$$P_{12\cdots L}^{(2s)} T_{0,12\cdots L}^{(1,1+)}\big(\lambda;\,\big\{w_j^{(2s;\epsilon)}\big\}_L\big) P_{12\cdots L}^{(2s)} = P_{12\cdots L}^{(2s)} T_{0,12\cdots L}^{(1,1+)}\big(\lambda;\,\big\{w_j^{(2s;\epsilon)}\big\}_L\big) + O(\epsilon), \tag{3.19}$$

$$P_{12\cdots L}^{(2s)} T_{0,12\cdots L}^{(1,1+)}\big(\lambda;\,\big\{w_j^{(2s;\epsilon)}\big\}_L\big) \widetilde{P}_{12\cdots L}^{(2s)} = P_{12\cdots L}^{(2s)} T_{0,12\cdots L}^{(1,1)}\big(\lambda;\,\big\{w_j^{(2s;\epsilon)}\big\}_L\big) + O(\epsilon). \tag{3.20}$$

For instance we have $P_{12\cdots L}^{(2s)} B^{(2s+;\epsilon)}(\lambda) P_{12\cdots L}^{(2s)} = P_{12\cdots L}^{(2s)} B^{(2s+;\epsilon)}(\lambda) + O(\epsilon)$.

Proof. Taking derivatives with respect to inhomogeneous parameters w_j, we can show

$$T_{0,12\cdots L}^{(1,1+)}\left(\lambda;\left\{w_j^{(2s+;\epsilon)}\right\}_L\right)=T_{0,12\cdots L}^{(1,1+)}\left(\lambda;\left\{w_j^{(2s)}\right\}_L\right)+O(\epsilon),\tag{3.21}$$

where $T_{0,12\cdots L}^{(1,1+)}(\lambda;\{w_j^{(2s)}\}_L)$ commutes with the projection operator $P_{12\cdots L}^{(2s)}$ as follows [15]:

$$P_{12\cdots L}^{(2s)}T_{0,12\cdots L}^{(1,1+)}\left(\lambda;\left\{w_j^{(2s)}\right\}_L\right)=P_{12\cdots L}^{(2s)}T_{0,12\cdots L}^{(1,1+)}\left(\lambda;\left\{w_j^{(2s)}\right\}_L\right)P_{12\cdots L}^{(2s)}.\tag{3.22}$$

We show (3.20) making use of (2.22). □

3.3. Expectation value of a local operator through the limit: $\epsilon\to 0$

In the massless regime, we define the expectation value of product of operators $\prod_{k=1}^{m}\widetilde{E}_k^{i_k,j_k(2s+)}$ with respect to an eigenstate $|\widetilde{\{\lambda_\alpha\}}_M^{(2s+)}\rangle$ by

$$\left\langle\prod_{k=1}^{m}\widetilde{E}_k^{i_k,j_k(2s+)}\right\rangle\left(\{\lambda_\alpha\}_M^{(2s+)}\right)=\frac{\langle\widetilde{\{\lambda_\alpha\}}_M^{(2s+)}|\prod_{k=1}^{m}\widetilde{E}_k^{i_k,j_k(2s+)}|\widetilde{\{\lambda_\alpha\}}_M^{(2s+)}\rangle}{\langle\widetilde{\{\lambda_\alpha\}}_M^{(2s+)}|\widetilde{\{\lambda_\alpha\}}_M^{(2s+)}\rangle}.\tag{3.23}$$

We evaluate the expectation value of a given spin-s local operator for a Bethe-ansatz eigenstate $|\{\lambda_\alpha\}_M^{(2s)}\rangle$, as follows. We first assume that the Bethe roots $\{\lambda_\alpha(\epsilon)\}_M$ are continuous with respect to small parameter ϵ. We express the spin-s local operator in terms of spin-1/2 global operators such as formula (3.18) with generic inhomogeneous parameters $w_j^{(2s;\epsilon)}$. Applying (3.19) and (3.20) we remove the projection operators out of the product of global operators. We next calculate the scalar product for the Bethe state $|\{\lambda_k(\epsilon)\}_M^{(2s;\epsilon)}\rangle$ which has the same inhomogeneous parameters $w_j^{(2s;\epsilon)}$, making use of the formulas of the spin-1/2 case. Then we take the limit of sending ϵ to 0, and obtain the expectation value of the spin-s local operator.

For an illustration, let us consider the expectation value of $\widetilde{E}_1^{n,n(2s+)}$. First, applying projection operator $P_{12\cdots L}^{(2s)}$ to $|\widetilde{\{\lambda_\alpha\}}_M^{(2s+)}\rangle=\prod_{\alpha=1}^{M}\widetilde{B}^{(2s+)}(\lambda_\alpha)|0\rangle$ we show

$$\begin{aligned}P_{1\cdots L}^{(2s)}\left|\widetilde{\{\lambda_\alpha\}}_M^{(2s+)}\right\rangle&=P_{1\cdots L}^{(2s)}\prod_{\alpha=1}^{M}B^{(2s+;\epsilon)}\left(\lambda_\alpha(\epsilon)\right)|0\rangle+O(\epsilon)\\&=e^{-\sum_{\alpha=1}^{M}\lambda_\alpha(\epsilon)}P_{1\cdots L}^{(2s)}\chi_{12\cdots L}\prod_{\alpha=1}^{M}B^{(2s;\epsilon)}\left(\lambda_\alpha(\epsilon)\right)|0\rangle+O(\epsilon).\end{aligned}\tag{3.24}$$

Second, making use of the relation $\langle 0|=\langle 0|P_{12\cdots L}^{(2s)}$, we show

$$\begin{aligned}\left\langle\widetilde{\{\lambda_\alpha\}}_M^{(2s+)}\right|&=\langle 0|\prod_{\alpha=1}^{M}C^{(2s+;\epsilon)}\left(\lambda_\alpha(\epsilon)\right)P_{1\cdots L}^{(2s)}+O(\epsilon)\\&=\langle 0|\prod_{\alpha=1}^{M}C^{(2s;\epsilon)}\left(\lambda_\alpha(\epsilon)\right)\chi_{12\cdots L}^{-1}P_{1\cdots L}^{(2s)}e^{\sum_{\alpha=1}^{M}\lambda_\alpha(\epsilon)}+O(\epsilon).\end{aligned}\tag{3.25}$$

Making use of (3.18) we have

$$
\begin{aligned}
&\big\langle \{\widetilde{\lambda_\alpha}\}_M^{(2s+)} \big| \widetilde{E}_1^{n\,n(2s+)} \big| \{\widetilde{\lambda_\alpha}\}_M^{(2s+)} \big\rangle \\
&\quad = \binom{2s}{n} \langle 0| \prod_{\alpha=1}^{M} C^{(2s+;\epsilon)}\big(\lambda_\alpha(\epsilon)\big) P_{1\cdots L}^{(2s)} \underline{\widetilde{P}_{12\cdots L}^{(2s)}} \prod_{k=1}^{n} D^{(2s+;\epsilon)}\big(w_k^{(2s;\epsilon)}\big) \\
&\qquad \times \prod_{k=n+1}^{2s} A^{(2s+;\epsilon)}\big(w_k^{(2s;\epsilon)}\big) \prod_{\alpha=2s+1}^{2sN_s} \big(A^{(2s+;\epsilon)} + D^{(2s+;\epsilon)}\big)\big(w_\alpha^{(2s;\epsilon)}\big) \underline{\widetilde{P}_{1\cdots L}^{(2s)}} \\
&\qquad \times \prod_{\alpha=1}^{M} \widetilde{B}^{(2s+)}(\lambda_\alpha)|0\rangle + O(\epsilon).
\end{aligned} \tag{3.26}
$$

Here we have $\prod_{j=1}^{2sN_s}(A^{(2s+;\epsilon)} + D^{(2s+;\epsilon)})(w_j^{(2s;\epsilon)}) = I^{\otimes L}$ for generic ϵ. We apply projection operators $P^{(2s)}$ to $\widetilde{P}^{(2s)}$ from the left, which are underlined in (3.26), and make use of (2.22). We then move the projection operators $P^{(2s)}$ in the leftward direction, making use of (3.19). Thus, the right-hand side of (3.26) is now given by the following:

$$
\begin{aligned}
&= \binom{2s}{n} \langle 0| \prod_{\alpha=1}^{M} C^{(2s+;\epsilon)}\big(\lambda_\alpha(\epsilon)\big) \prod_{k=1}^{n} D^{(2s+;\epsilon)}\big(w_k^{(2s;\epsilon)}\big) \prod_{k=n+1}^{2s} A^{(2s+;\epsilon)}\big(w_k^{(2s;\epsilon)}\big) \\
&\quad \times \prod_{j=2s+1}^{2sN_s} \big(A^{(2s+;\epsilon)} + D^{(2s+;\epsilon)}\big)\big(w_j^{(2s;\epsilon)}\big) \prod_{\beta=1}^{M} B^{(2s+;\epsilon)}\big(\lambda_\beta(\epsilon)\big)|0\rangle + O(\epsilon).
\end{aligned} \tag{3.27}
$$

After applying the gauge transformation $\chi_{1\cdots L}^{-1}$ inverse to (2.7) [15], we obtain

$$
\begin{aligned}
&\big\langle \{\widetilde{\lambda_\alpha}\}_M^{(2s+)} \big| \widetilde{E}_1^{n\,n(2s+)} \big| \{\widetilde{\lambda_\alpha}\}_M^{(2s+)} \big\rangle \\
&\quad = \binom{2s}{n} \lim_{\epsilon\to 0} \langle 0| \prod_{\alpha=1}^{M} C^{(2s;\epsilon)}\big(\lambda_\alpha(\epsilon)\big) \prod_{k=1}^{n} D^{(2s;\epsilon)}\big(w_k^{(2s;\epsilon)}\big) \prod_{k=n+1}^{2s} A^{(2s;\epsilon)}\big(w_k^{(2s;\epsilon)}\big) \\
&\qquad \times \prod_{j=2s+1}^{2sN_s} \big(A^{(2s;\epsilon)} + D^{(2s;\epsilon)}\big)\big(w_j^{(2s;\epsilon)}\big) \prod_{\beta=1}^{M} B^{(2s;\epsilon)}\big(\lambda_\beta(\epsilon)\big)|0\rangle.
\end{aligned} \tag{3.28}
$$

Here $A^{(2s;\epsilon)}$ and $D^{(2s;\epsilon)}$ denote matrix elements $A^{(2s\,p;\epsilon)}$ and $D^{(2s\,p;\epsilon)}$ of the monodromy matrix with principal grading, respectively. In the last line of (3.27), we have evaluated the eigenvalue of transfer matrix $A^{(2s;\epsilon)}(w_j^{(2s;\epsilon)}) + D^{(2s;\epsilon)}(w_j^{(2s;\epsilon)})$ on the eigenstate $|\{\lambda_\beta(\epsilon)\}_M^{(2s;\epsilon)}\rangle$ as follows:

$$
\begin{aligned}
&\prod_{j=2s+1}^{2sN_s} \big(A^{(2s;\epsilon)}\big(w_j^{(2s;\epsilon)}\big) + D^{(2s;\epsilon)}\big(w_j^{(2s;\epsilon)}\big)\big) \big| \big\{\lambda_\beta(\epsilon)\big\}_M^{(2s;\epsilon)} \big\rangle \\
&\quad = \left(\prod_{j=2s+1}^{2sN_s} \prod_{\alpha=1}^{M} b^{-1}\big(\lambda_\alpha(\epsilon) - w_j^{(2s;\epsilon)}\big) \right) \big| \big\{\lambda_\beta(\epsilon)\big\}_M^{(2s;\epsilon)} \big\rangle.
\end{aligned} \tag{3.29}
$$

Before sending ϵ to 0, we expand the products of C operators multiplied by operators A and D by the commutation relations between C and A as well as C and D, respectively. We then evaluate the scalar product of B and C operators with inhomogeneous parameters $w_j^{(2s;\epsilon)}$. Finally, we derive the expectation value in the limit of sending ϵ to 0.

Sending ϵ to 0, we calculate the expectation value of $A^{(2s)}(\lambda)+D^{(2s)}(\lambda)$ at $\lambda=w_2^{(2s)}$. For instance, we calculate $A^{(2s;\epsilon)}(w_2^{(2s;\epsilon)})+D^{(2s;\epsilon)}(w_2^{(2s;\epsilon)})$ on the vacuum $|0\rangle$ as follows:

$$\begin{aligned}
&\lim_{\epsilon\to 0}\langle 0|\big(A^{(2s;\epsilon)}\big(w_2^{(2s;\epsilon)}\big)+D^{(2s;\epsilon)}\big(w_2^{(2s;\epsilon)}\big)\big)|0\rangle \\
&\quad=\lim_{\epsilon\to 0}\langle 0|\big(A\big(w_2^{(2s;\epsilon)};\big\{w_j^{(2s;\epsilon)}\big\}_L\big)+D^{(2s)}\big(w_2^{(2s;\epsilon)};\big\{w_j^{(2s;\epsilon)}\big\}_L\big)\big)|0\rangle \\
&\quad=\lim_{\epsilon\to 0}\left(1+\prod_{j=1}^{\ell N_s} b\big(w_2^{(2s;\epsilon)}-w_j^{(2s;\epsilon)}\big)\right)\langle 0|0\rangle \\
&\quad=(1+0)\langle 0|0\rangle.
\end{aligned}\tag{3.30}$$

If we put $\lambda=w_2^{(2s)}$ after sending ϵ to 0, the result is different from (3.30) as follows:

$$\begin{aligned}
&\lim_{\lambda\to w_2^{(2s)}}\langle 0|\big(A^{(2s)}\big(\lambda;\big\{w_j^{(2s)}\big\}_L\big)+D^{(2s)}\big(\lambda;\big\{w_j^{(2s)}\big\}_L\big)\big)|0\rangle \\
&\quad=\left(1+\prod_{p=1}^{N_s} b_\ell\big(w_2^{(2s)}-\xi_p\big)\right)\langle 0|0\rangle.
\end{aligned}\tag{3.31}$$

4. Derivation of matrix S

4.1. The ground-state solution of 2s-strings

We shall introduce ℓ-strings for an integer ℓ. Let us shift rapidities λ_j by $s\eta$ such as $\tilde{\lambda}_j=\lambda_j+s\eta$. Then, the Bethe-ansatz equations (3.9) are given by

$$\prod_{p=1}^{N_s}\frac{\sinh(\tilde{\lambda}_j-\xi_p+s\eta)}{\sinh(\tilde{\lambda}_j-\xi_p-s\eta)}=\prod_{\beta=1;\,\beta\neq\alpha}^{n}\frac{\sinh(\tilde{\lambda}_j-\tilde{\lambda}_\beta+\eta)}{\sinh(\tilde{\lambda}_j-\tilde{\lambda}_\beta-\eta)},\quad \text{for } j=1,2,\ldots,n.\tag{4.1}$$

We define an ℓ-string by the following set of rapidities

$$\tilde{\lambda}_a^{(\alpha)}=\mu_a+(\ell+1-2\alpha)\frac{\eta}{2}+\epsilon_a^{(\alpha)}\quad \text{for } \alpha=1,2,\ldots,\ell.\tag{4.2}$$

We call μ_a the center of the ℓ-string and $\epsilon_a^{(\alpha)}$ string deviations. We assume that $\epsilon_a^{(\alpha)}$ are very small for large N_s:

$$\lim_{N_s\to\infty}\epsilon_a^{(\alpha)}=0.\tag{4.3}$$

If they are zero, then we call the set of rapidities of (4.2) a complete ℓ-string. The string center μ_a corresponds to the central position among the ℓ complex numbers: $\tilde{\lambda}_a^{(1)},\tilde{\lambda}_a^{(2)},\ldots,\tilde{\lambda}_a^{(\ell)}$. Furthermore we assume that μ_a are real. If inhomogeneous parameters, ξ_p, are small enough, then the Bethe-ansatz equations should have an ℓ-strings as a solution.

In terms of rapidities λ_j which are not shifted, an ℓ-string is expressed in the following form:

$$\lambda_a^{(\alpha)}=\mu_a-(\alpha-1/2)\eta+\epsilon_a^{(\alpha)}\quad \text{for } \alpha=1,2,\ldots,\ell.\tag{4.4}$$

We denote $\lambda_a^{(\alpha)}$ also by $\lambda_{(a,\alpha)}$.

Let us now introduce the conjecture that the ground state of the spin-s case $|\psi_g^{(2s)}\rangle$ is given by $N_s/2$ sets of $2s$-strings:

$$\lambda_a^{(\alpha)} = \mu_a - (\alpha - 1/2)\eta + \epsilon_a^{(\alpha)}, \quad \text{for } a = 1, 2, \ldots, N_s/2 \text{ and } \alpha = 1, 2, \ldots, 2s. \tag{4.5}$$

In terms of $\lambda_a^{(\alpha)}$s in the massless regime, for $w = +$ and p, we have

$$\left|\psi_g^{(2s\,w)}\right\rangle = \prod_{a=1}^{N_s/2} \prod_{\alpha=1}^{2s} \widetilde{B}^{(2s\,w)}\left(\lambda_a^{(\alpha)}; \{\xi_p\}\right)|0\rangle. \tag{4.6}$$

Hereafter we set $M = 2sN_s/2 = sN_s$.

According to analytic and numerical studies [40–42], we may assume the following properties of string deviations $\epsilon_a^{(\alpha)}$s. When N_s is very large, the deviations are given by

$$\epsilon_a^{(\alpha)} = i\delta_a^{(\alpha)}, \tag{4.7}$$

where i denotes $\sqrt{-1}$, and $\delta_a^{(\alpha)}$ are real. Moreover, $\delta_a^{(\alpha)} - \delta_a^{(\alpha+1)} > 0$ for $\alpha = 1, 2, \ldots, 2s - 1$, and $|\delta_a^{(\alpha)}| > |\delta_a^{(\alpha+1)}|$ for $\alpha < s$, while $|\delta_a^{(\alpha)}| < |\delta_a^{(\alpha+1)}|$ for $\alpha \geqslant s$.

In the thermodynamic limit: $N_s \to \infty$, the Bethe-ansatz equations for the ground state of the higher-spin XXZ chain become the integral equation for the string centers, as shown in Appendix F [53]. The density of string centers, $\rho_{\rm tot}(\mu)$, is given by

$$\rho_{\rm tot}(\mu) = \frac{1}{N_s} \sum_{p=1}^{N_s} \frac{1}{2\zeta \cosh(\pi(\mu - \xi_p)/\zeta)} \tag{4.8}$$

Thus, the sum over all the Bethe roots of the ground state is evaluated by integrals in the thermodynamic limit, $N_s \to \infty$, as follows:

$$\begin{aligned} \frac{1}{N_s} \sum_{A=1}^{M} f(\lambda_A) &= \frac{1}{N_s} \sum_{\alpha=1}^{2s} \sum_{a=1}^{N_s/2} f(\lambda_{(a,\alpha)}) \\ &= \sum_{\alpha=1}^{2s} \int_{-\infty}^{\infty} f\left(\mu_a - (\alpha - 1/2)\eta + \epsilon_a^{(\alpha)}\right) \rho_{\rm tot}(\mu_a)\, d\mu_a + O(1/N_s). \end{aligned} \tag{4.9}$$

For the homogeneous chain where $\xi_p = 0$ for $p = 1, 2, \ldots, N_s$, we denote the density of string centers by $\rho(\lambda)$

$$\rho(\lambda) = \frac{1}{2\zeta \cosh(\pi\lambda/\zeta)}. \tag{4.10}$$

Let us introduce useful notation of the suffix of rapidities. For rapidities $\lambda_a^{(\alpha)} = \lambda_{(a,\alpha)}$ we define integers A by $A = 2s(a - 1) + \alpha$ for $a = 1, 2, \ldots, N_s/2$ and for $\alpha = 1, 2, \ldots, 2s$. We thus denote $\lambda_{(a,\alpha)}$ also by λ_A for $A = 1, 2, \ldots, sN_s$, and put $\lambda_{(a,\alpha)}$ in increasing order with respect to $A = 2s(a - 1) + \alpha$ such as $\lambda_{(1,1)} = \lambda_1, \lambda_{(1,2)} = \lambda_2, \ldots, \lambda_{(N_s/2,2s)} = \lambda_{sN_s}$.

In the ground state rapidities λ_A for $A = 1, 2, \ldots, M$, are now expressed by

$$\lambda_{2s(a-1)+\alpha} = \mu_a - (\alpha - 1/2)\eta + \epsilon_a^{(\alpha)} \quad \text{for } a = 1, 2, \ldots, N_s/2 \text{ and } \alpha = 1, 2, \ldots, 2s. \tag{4.11}$$

For a given real number x, let us denote by $[x]$ the greatest integer less than or equal to x. When $A = 2s(a - 1) + \alpha$ with $1 \leqslant \alpha \leqslant 2s$, integer a is given by $a = [(A - 1)/2s] + 1$, and integer α is given by $\alpha = A - 2s[(A - 1)/2s]$.

4.2. Derivation of the spin-s EFP for a finite chain

We define the emptiness formation probability (EFP) for the spin-s case by

$$\tau^{(2s+)}(m) = \frac{\langle \psi_g^{(2s+)} | \widetilde{E}_1^{2s,2s(2s+)} \cdots \widetilde{E}_m^{2s,2s(2s+)} | \psi_g^{(2s+)} \rangle}{\langle \psi_g^{(2s+)} | \psi_g^{(2s+)} \rangle}. \tag{4.12}$$

We shall denote $\tau^{(2s+)}(m)$ by $\tau^{(2s)}(m)$.

Let us assume that Bethe roots $\{\lambda_\alpha(\epsilon)\}_M$ with inhomogeneous parameters $w_j^{(2s;\epsilon)}$ ($j = 1, 2, \ldots, L$; $L = 2sN_s$) become the ground-state solution of the spin-s XXZ spin chain, $\{\lambda_\alpha\}_M$, in the limit of sending ϵ to 0. We denote the Bethe vector with Bethe roots $\{\lambda_\alpha(\epsilon)\}_M$ by

$$\begin{aligned}
\left|\psi_g^{(2s+;\epsilon)}\right\rangle &= \prod_{\alpha=1}^{M} B^{(2s;\epsilon)}\bigl(\lambda_\alpha(\epsilon)\bigr)|0\rangle = e^{-\sum_{\alpha=1}^{M}\lambda_\alpha(\epsilon)}\chi_{12\cdots L} \cdot \prod_{\alpha=1}^{M} B^{(2s\,p;\epsilon)}\bigl(\lambda_\alpha(\epsilon)\bigr)|0\rangle \\
&= e^{-\sum_{\alpha=1}^{M}\lambda_\alpha(\epsilon)}\chi_{12\cdots L}\left|\psi_g^{(2s;\epsilon)}\right\rangle.
\end{aligned} \tag{4.13}$$

Here we recall the transformation inverse to (2.7). We now calculate the norm of the spin-s ground state from that of the spin-1/2 case through the limit of sending ϵ to 0 as follows:

$$\begin{aligned}
&\left\langle\psi_g^{(2s+)}\middle|\psi_g^{(2s+)}\right\rangle \\
&\quad = \lim_{\epsilon\to 0}\left\langle\psi_g^{(2s;\epsilon)}\middle|\psi_g^{(2s;\epsilon)}\right\rangle \\
&\quad = \lim_{\epsilon\to 0}\langle 0| \prod_{k=1}^{M} C^{(2s;\epsilon)}(\lambda_k) \prod_{j=1}^{M} B^{(2s;\epsilon)}(\lambda_j)|0\rangle \\
&\quad = \lim_{\epsilon\to 0}\sinh^M\eta \prod_{j,k=1;\, j\neq k}^{M} b^{-1}\bigl(\lambda_j(\epsilon),\lambda_k(\epsilon)\bigr)\cdot \det\Phi^{(1)'}\left(\bigl\{\lambda_k(\epsilon)\bigr\}_M; \bigl\{w_j^{(2s;\epsilon)}\bigr\}_L\right) \\
&\quad = \sinh^M\eta \prod_{j,k=1;\, j\neq k}^{M} b^{-1}(\lambda_j,\lambda_k)\cdot \det\Phi^{(2s)'}\bigl(\{\lambda_k\}_M; \{\xi_p\}_{N_s}\bigr)
\end{aligned} \tag{4.14}$$

where matrix elements of the spin-s Gaudin matrix for $j, k = 1, 2, \ldots, M$, are given by

$$\begin{aligned}
\Phi_{j,k}^{(2s)'}\bigl(\{\lambda_l\}_M; \{\xi_p\}\bigr) &= -\frac{\partial}{\partial\lambda_k}\log\left(\frac{a^{(2s)}(\lambda_j)}{d^{(2s)}(\lambda_j)}\prod_{t\neq j}\frac{\sinh(\lambda_t-\lambda_j+\eta)}{\sinh(\lambda_t-\lambda_j-\eta)}\right) \\
&= \delta_{j,k}\left(\sum_{p=1}^{N_s}\frac{\sinh(2s\eta)}{\sinh(\lambda_j-\xi_p)\sinh(\lambda_j-\xi_p+2s\eta)}\right. \\
&\qquad \left. -\sum_{C=1}^{M}\frac{\sinh 2\eta}{\sinh(\lambda_j-\lambda_C+\eta)\sinh(\lambda_j-\lambda_C-\eta)}\right) \\
&\qquad + \frac{\sinh 2\eta}{\sinh(\lambda_j-\lambda_k+\eta)\sinh(\lambda_j-\lambda_k-\eta)}.
\end{aligned} \tag{4.15}$$

By applying formula (3.18) with $n = 2s$, the numerator of (4.12) is given by

$$
\begin{aligned}
&\big\langle \psi_g^{(2s+;\epsilon)} \big| \widetilde{E}_1^{2s,2s(2s+)} \cdots \widetilde{E}_m^{2s,2s(2s+)} \big| \psi_g^{(2s+;\epsilon)} \big\rangle = \lim_{\epsilon\to 0} \big\langle \psi_g^{(2s;\epsilon)} \big| \prod_{k=1}^{m} E_k^{2s,2s(2s)} \big| \psi_g^{(2s;\epsilon)} \big\rangle \\
&\quad = \lim_{\epsilon\to 0} \big\langle \psi_g^{(2s;\epsilon)} \big| P_{12\cdots L}^{(2s)} \prod_{i=1}^{m} \Bigg(\prod_{\alpha=1}^{2s(i-1)} \big(A^{(2s;\epsilon)} + D^{(2s;\epsilon)}\big)\big(w_\alpha^{(2s;\epsilon)}\big) \cdot \prod_{k=1}^{2s} D^{(2s;\epsilon)}\big(w_{2s(i-1)+k}^{(2s;\epsilon)}\big) \\
&\qquad \times \prod_{\alpha=1}^{2sN_s} \big(A^{(2s;\epsilon)} + D^{(2s;\epsilon)}\big)\big(w_\alpha^{(2s;\epsilon)}\big) \Bigg) P_{12\cdots L}^{(2s)} \big| \psi_g^{(2s)} \big\rangle \\
&\quad = \prod_{j=1}^{m} \prod_{\alpha=1}^{M} b_{2s}(\lambda_\alpha, \xi_j) \lim_{\epsilon\to 0} \big\langle \psi_g^{(2s;\epsilon)} \big| D^{(2s;\epsilon)}\big(w_1^{(2s;\epsilon)}\big) \cdots D^{(2s;\epsilon)}\big(w_{2sm}^{(2s;\epsilon)}\big) \big| \psi_g^{(2s;\epsilon)} \big\rangle. \qquad (4.16)
\end{aligned}
$$

Let us set $\lambda_{M+j}(\epsilon) = w_j^{(2s;\epsilon)}$ for $j = 1, 2, \ldots, 2sm$. Applying formula (G.1) to (4.16) we have

$$
\begin{aligned}
&\langle 0| \prod_{\alpha=1}^{M} C^{(2s;\epsilon)}\big(\lambda_\alpha(\epsilon)\big) \prod_{j=1}^{2sm} D^{(2s;\epsilon)}\big(\lambda_{M+j}(\epsilon)\big) \prod_{\beta=1}^{M} B^{(2s;\epsilon)}\big(\lambda_\beta(\epsilon)\big) |0\rangle \\
&\quad = \sum_{c_1=1}^{M} \sum_{c_2=1;\, c_2\neq c_1}^{M} \cdots \sum_{\substack{c_{2sm}=1;\\ c_{2sm}\neq c_1,\ldots,c_{2sm-1}}}^{M} G_{c_1\cdots c_{2sm}}\big(\lambda_1(\epsilon), \ldots, \lambda_{M+2sm}(\epsilon); \big\{w_j^{(2s;\epsilon)}\big\}_L\big) \\
&\qquad \times \langle 0| \prod_{k=1;\, k\neq c_1,\ldots,c_{2sm}}^{M+2sm} C^{(2s;\epsilon)}\big(\lambda_k(\epsilon)\big) \prod_{\alpha=1}^{M} B^{(2s;\epsilon)}\big(w_j^{(2s;\epsilon)}\big) |0\rangle, \qquad (4.17)
\end{aligned}
$$

where

$$
\begin{aligned}
&G_{c_1\cdots c_{2sm}}\big(\lambda_1, \ldots, \lambda_{M+2sm}; \{w_j\}_L\big) \\
&\quad = \prod_{j=1}^{2sm} \left(d\big(\lambda_{c_j}; \{w_j\}_L\big) \frac{\prod_{t=1;\, t\neq c_1,\ldots,c_{j-1}}^{M+j-1} \sinh(\lambda_{c_j} - \lambda_t + \eta)}{\prod_{t=1;\, t\neq c_1,\ldots,c_j}^{M+j} \sinh(\lambda_{c_j} - \lambda_t)} \right). \qquad (4.18)
\end{aligned}
$$

We remark that from (4.18) the set of integers $c_1, \ldots, c_{2sm}$ of the most dominant terms in (4.17) are given by m sets of $2s$-strings. If they are not, the numerator of (4.18) and hence the right-hand side of (4.17) becomes smaller at least by the order of $1/N_s$ in the large N_s limit. However, each of the most dominant terms diverges with respect to N_s in the large-N_s limit, and they should cancel each other so that the final result becomes finite. We therefore calculate all possible contributions with respect to the set of integers, $c_1, c_2, \ldots, c_{2sm}$.

Let us take a sequence of distinct integers c_j satisfying $1 \leqslant c_j \leqslant M$ for $j = 1, 2, \ldots, 2sm$. We denote it by $(c_j)_{2sm}$, i.e. $(c_j)_{2sm} = (c_1, c_2, \ldots, c_{2sm})$. Let us denote by Σ_M the set of integers, $1, 2, \ldots, M$: $\Sigma_M = \{1, 2, \ldots, M\}$. We then consider the complementary set of integers $\Sigma_M \setminus \{c_1, \ldots, c_{2sm}\}$, and put the elements in increasing order such as $z_1 < z_2 < \cdots < z_{M-2sm}$. We then extend the sequence z_n of $M - 2sm$ integers into that of M integers by setting $z_{j+M-2sm} = c_j$ for $j = 1, 2, \ldots, 2sm$. We shall denote z_n also by $z(n)$ for $n = 1, 2, \ldots, M$.

In terms of sequence $(z_n)_M$ we express the scalar product in the last line of (4.17) as follows:

$$
\begin{aligned}
&\langle 0| \prod_{k=1;\, k\neq c_1,\ldots,c_{2sm}}^{M+2sm} C^{(2s;\epsilon)}\big(\lambda_k(\epsilon)\big) \prod_{\alpha=1}^{M} B^{(2s;\epsilon)}\big(\lambda_\alpha(\epsilon)\big)|0\rangle \\
&\quad = \langle 0| \prod_{k=1}^{M-2sm} C^{(2s;\epsilon)}\big(\lambda_{z(k)}(\epsilon)\big) \prod_{j=1}^{2sm} C^{(2s;\epsilon)}\big(w_j^{(2s;\epsilon)}\big) \\
&\qquad \times \prod_{i=1}^{M-2sm} B^{(2s;\epsilon)}\big(\lambda_{z(i)}(\epsilon)\big) \prod_{j=1}^{2sm} B^{(2s;\epsilon)}\big(\lambda_{c_j}(\epsilon)\big)|0\rangle.
\end{aligned} \tag{4.19}
$$

We evaluate scalar product (4.19), sending ν_j to $\lambda_{z(j)}(\epsilon)$ for $j \leqslant M-2sm$ and to $w_{j-M+2sm}^{(2s;\epsilon)}$ for $j > M-2sm$ in the following matrix:

$$
H^{(1)}\Big(\big(\lambda_{z(k)}(\epsilon)\big)_M, (\nu_{z(1)}, \ldots, \nu_{z(M-2sm)}, \nu_{M-2sm+1}, \ldots, \nu_M); \big(w_j^{(2s;\epsilon)}\big)_L\Big). \tag{4.20}
$$

Here we define the matrix elements $H_{ab}^{(2s)}(\{\lambda_\alpha\}_n, \{\mu_j\}_n; \{\xi_k\}_{N_s})$ for $a, b = 1, 2, \ldots, n$, by

$$
\begin{aligned}
&H_{ab}^{(2s)}\big(\{\lambda_\alpha\}_n, \{\mu_j\}_n; \{\xi_k\}_{N_s}\big) \\
&\quad = \frac{\sinh\eta}{\sinh(\lambda_a-\mu_b)} \\
&\qquad \times \left(\frac{a(\mu_b)}{d^{(2s)}(\mu_b;\{\xi_k\})} \prod_{k=1;\, k\neq a}^{n} \sinh(\lambda_k-\mu_b+\eta) - \prod_{K=1;\, k\neq a}^{n} \sinh(\lambda_k-\mu_b-\eta) \right).
\end{aligned} \tag{4.21}
$$

Let us denote $M-2sm$ by M'. We write the composite of two sequences $(a(i))_M$ and $(b(j))_N$ as $(a(i))_M \# (b(j))_N$. Explicitly we have

$$
\big(a(i)\big)_M \# \big(b(j)\big)_N = \big(a(1), \ldots, a(M), b(1), \ldots, b(N)\big). \tag{4.22}
$$

For $j > M' = M-2sm$, we have

$$
\begin{aligned}
&\lim_{\nu_j \to w_{j-M'}^{(2s;\epsilon)}} d\big(\nu_j; \{w_j^{(2s;\epsilon)}\}_L\big) H_{i,j}^{(1)}\Big(\big(\lambda_{z(k)}(\epsilon)\big)_M, (\nu_k)_{M'} \# (\nu_{k+M'})_{2sm}; \big(w_j^{(2s;\epsilon)}\big)_L\Big) \\
&\quad = \prod_{\alpha=1}^{M} \sinh\big(\lambda_\alpha(\epsilon) - w_{j-M'}^{(2s;\epsilon)} + \eta\big) \left(\frac{\sinh\eta}{\sinh(\lambda_{z(i)}(\epsilon) - w_{j-M'}^{(2s;\epsilon)}) \sinh(\lambda_{z(i)}(\epsilon) - w_{j-M'}^{(2s;\epsilon)} + \eta)} \right. \\
&\qquad - d\big(w_{j-M'}^{(2s;\epsilon)}; \{w_j^{(2s;\epsilon)}\}_L\big) \prod_{t=1}^{M} \frac{\sinh(\lambda_t(\epsilon) - w_{j-M'}^{(2s;\epsilon)} - \eta)}{\sinh(\lambda_t(\epsilon) - w_{j-M'}^{(2s;\epsilon)} + \eta)} \\
&\qquad \left. \times \frac{\sinh\eta}{\sinh(\lambda_{z(i)}(\epsilon) - w_{j-M'-1}^{(2s;\epsilon)}) \sinh(\lambda_{z(i)}(\epsilon) - w_{j-M'-1}^{(2s;\epsilon)} + \eta)} \right).
\end{aligned} \tag{4.23}
$$

The second term of (4.23) for matrix element (i, j) vanishes since we have $d(w_{j-M'}^{(2s;\epsilon)}; \{w_k^{(2s;\epsilon)}\}_L) = 0$. Here we remark that if we directly evaluate matrix $H^{(2s)}$ at $\epsilon = 0$, the second term of (4.23) for matrix element (i, j) for $j \neq 2s(n-1)+1+M'$ with $n = 1, 2, \ldots, m$, does not vanish, although it is deleted by subtracting column j by column $j-1$, as discussed for the XXX case in Ref. [13]. We thus have

$$\begin{aligned}&\lim_{\epsilon\to 0}\det H^{(1)}\big(\big(\lambda_{z(k)}(\epsilon)\big)_M, \big(\lambda_{z(1)}(\epsilon),\ldots,\lambda_{z(M-2sm)}(\epsilon), w_1^{(2s;\epsilon)},\ldots,w_{2sm}^{(2s;\epsilon)}\big);\\ &\quad \big(w_j^{(2s;\epsilon)}\big)_{2sN_s}\big)\\ &\quad = (-1)^{M-2sm}\prod_{b=1}^{M-2sm}\prod_{k=1}^{M}\sinh(\lambda_k-\lambda_{z_b}-\eta)\prod_{j=1}^{2sm}\prod_{k=1}^{M}\sinh\big(\lambda_k-w_j^{(2s)}+\eta\big)\\ &\qquad\times\det\Psi^{(2s)'}\big((\lambda_{z(i)})_M, (\lambda_{z(i)})_{M'}\#\big(w_j^{(2s)}\big)_{2sm};\{\xi_p\}_{N_s}\big)\end{aligned}\tag{4.24}$$

where (i,j) element of $\Psi^{(2s)'}((\lambda_{z(k)})_M, (\lambda_{z(k)})_{M'}\#(w_k^{(2s)})_{2sm};\{\xi_p\}_{N_s})$ for $i=1,2,\ldots,M$, are given by

$$\begin{aligned}&\Psi_{i,j}^{(2s)'}\big((\lambda_{z(1)},\ldots,\lambda_{z(M-2sm)},\lambda_{c_1},\ldots,\lambda_{c_{2sm}}), \big(\lambda_{z(1)},\ldots,\lambda_{z(M-2sm)}, w_1^{(2s)},\ldots,w_{2sm}^{(2s)}\big);\\ &\quad (\xi_p)_{N_s}\big)\\ &\quad = \begin{cases}\Phi_{z(i),z(j)}^{(2s)'}((\lambda_k)_M;\{\xi_p\}) & \text{for } j\leqslant M-2sm,\\ \dfrac{\sinh\eta}{\sinh(\lambda_{z(i)}-w_{j-M'}^{(2s)})\sinh(\lambda_{z(i)}-w_{j-M'}^{(2s)}+\eta)} & \text{for } j> M-2sm.\end{cases}\end{aligned}\tag{4.25}$$

Therefore, for $i,j=1,2,\ldots,M-2sm$, we have

$$\big(\big(\Phi^{(2s)'}\big((\lambda_{z(k)})_M;\{\xi_p\}\big)\big)^{-1}\Psi^{(2s)'}\big((\lambda_{z(k)})_M, (\lambda_{z(k)})_{M'}\#\big(w_j^{(2s)}\big)_{2sm};\{\xi_p\}\big)\big)_{i,j}=\delta_{i,j}.\tag{4.26}$$

In terms of sequence $(c_j)_{2sm}$, we express the dependence of matrix $(\Phi^{(2s)'})^{-1}\Psi^{(2s)'}$ on the sequence of Bethe roots $(\lambda_{z(i)})_M$, etc., briefly, as follows:

$$\begin{aligned}&\big(\Phi^{(2s)'}\big)^{-1}\Psi^{(2s)'}\big((c_j)_{2sm},\{\xi_p\}\big)\\ &\quad=\big(\Phi^{(2s)'}\big)^{-1}\Psi^{(2s)'}\big((\lambda_{z(j)})_M, (\lambda_{z(j)})_{M'}\#\big(w_j^{(2s)}\big)_{2sm};\{\xi_p\}\big).\end{aligned}\tag{4.27}$$

Recall $M'=M-2sm$. Similarly, we define $\Phi^{(2s)'}((c_j)_{2sm},\{\xi_p\})$ and $\Psi^{(2s)'}((c_j)_{2sm},\{\xi_p\})$ by

$$\begin{aligned}&\Phi_{i,j}^{(2s)'}\big((c_l)_{2sm},\{\xi_p\}\big)=\Phi_{i,j}^{(2s)'}\big(\{\lambda_{z(k)}\}_M;(\xi_p)_{N_s}\big)=\Phi_{z(i),z(j)}^{(2s)'}\big(\{\lambda_k\}_M;(\xi_p)_{N_s}\big),\\ &\Psi_{i,j}^{(2s)'}\big((c_k)_{2sm},\{\xi_p\}\big)=\Psi_{i,j}^{(2s)'}\big((\lambda_{z(k)})_M, (\lambda_{z(k)})_{M'}\#\big(w_k^{(2s)}\big)_{2sm};(\xi_p)_{N_s}\big),\end{aligned}\tag{4.28}$$

for $i,j=1,2,\ldots,M$. Here we remark again that sequence $(z(i))_M$ is determined by sequence $(c_1,\ldots,c_{2sm})$ by the definition that $\{z(1),z(2),\ldots,z(M')\}=\{1,2,\ldots,M\}\setminus\{c_1,\ldots,c_{2sm}\}$, and $z(1)<\cdots<z(M')$ while $z(j+M')=c_j$ for $j=1,2,\ldots,2sm$.

From property (4.26) we define a $2sm$-by-$2sm$ matrix $\phi^{(2s;m)}((c_j)_{2sm};\{\xi_p\})$ by

$$\begin{aligned}&\phi^{(2s;m)}\big((c_j)_{2sm};\{\xi_p\}\big)_{j,k}=\big(\big(\Phi^{(2s)'}\big)^{-1}\Psi^{(2s)'}\big((c_j)_{2sm};\{\xi_p\}\big)\big)_{j+M',k+M'}\\ &\quad\text{for } j,k=1,2,\ldots,2sm.\end{aligned}\tag{4.29}$$

Making use of (G.2), we obtain the spin-s EFP for the finite-size chain as follows:

$$\tau_{N_s}^{(2s)}(m)$$
$$= \frac{1}{\prod_{1\leqslant j<r\leqslant 2s}\sinh^m(r-j)\eta} \times \frac{1}{\prod_{1\leqslant k<l\leqslant m}\prod_{j=1}^{2s}\prod_{r=1}^{2s}\sinh(\xi_k-\xi_l+(r-j)\eta)}$$
$$\times \sum_{c_1=1}^{M}\sum_{c_2=1;\,c_2\neq c_1}^{M}\cdots$$
$$\times \sum_{\substack{c_{2sm}=1;\\ c_{2sm}\neq c_1,\dots,c_{2sm-1}}}^{M} H^{(2s)}\big(\lambda_{c_1},\dots,\lambda_{c_{2sm}};\{\xi_p\}\big)\det\big(\phi^{(2s;m)}\big((c_j)_{2sm};\{\xi_p\}\big)\big) \tag{4.30}$$

where $H^{(2s)}(\lambda_{c_1},\dots,\lambda_{c_{2sm}};\{\xi_p\})$ is given by

$$H^{(2s)}\big(\lambda_{c_1},\dots,\lambda_{c_{2sm}};\{\xi_p\}\big)$$
$$= \frac{1}{\prod_{1\leqslant l<k\leqslant 2sm}\sinh(\lambda_{c_k}-\lambda_{c_l}+\eta)} \times \prod_{j=1}^{2sm}\prod_{b=1}^{m}\prod_{\beta=1}^{2s-1}\sinh(\lambda_{c_j}-\xi_b+\beta\eta)$$
$$\times \prod_{l=1}^{m}\prod_{r_l=1}^{2s}\left(\prod_{b=1}^{l-1}\sinh(\lambda_{c_{2s(l-1)+r_l}}-\xi_b+2s\eta)\prod_{k=l+1}^{m}\sinh(\lambda_{c_{2s(l-1)+r_l}}-\xi_b)\right). \tag{4.31}$$

4.3. Diagonal elements of the spin-s Gaudin matrix

Let us define $K_n(\lambda)$ for $\eta=i\zeta$ with $0<\zeta<\pi$ by

$$K_n(\lambda)=\frac{1}{2\pi i}\frac{\sinh(n\eta)}{\sinh(\lambda-n\eta/2)\sinh(\lambda+n\eta/2)}. \tag{4.32}$$

Lemma 4.1. *For $0<\zeta<\pi/2s$, we have*

$$K_1(\lambda+n\eta)=\int_{-\infty}^{\infty}K_2(\lambda-\mu+n\eta+i\epsilon)\rho(\mu)\,d\mu \quad (n=1,2,\dots,2s-1), \tag{4.33}$$

and

$$K_1(\lambda-n\eta)=\int_{-\infty}^{\infty}K_2(\lambda-\mu-n\eta-i\epsilon)\rho(\mu)\,d\mu \quad (n=1,2,\dots,2s-1). \tag{4.34}$$

Here we recall $\eta=i\zeta$.

Proof. We first consider the case of positive n. Let us recall the Lieb equation

$$\rho(\lambda)=K_1(\lambda)-\int_{-\infty}^{\infty}K_2(\lambda-\mu)\rho(\mu)\,d\mu. \tag{4.35}$$

Shifting variable λ analytically to $\lambda+i\zeta-i\epsilon$ in (4.35) we have

$$\rho(\lambda + i\zeta - i\epsilon) = K_1(\lambda + i\zeta - i\epsilon) - \int_{-\infty}^{\infty} K_2(\lambda - \mu + i\zeta - i\epsilon)\rho(\mu)\,d\mu. \tag{4.36}$$

Using

$$\frac{1}{\sinh(\lambda - \mu - i\epsilon)} = \frac{1}{\sinh(\lambda - \mu + i\epsilon)} + 2\pi i\delta(\mu - \lambda) \tag{4.37}$$

we have

$$\int_{-\infty}^{\infty} K_2(\lambda - \mu + i\zeta - i\epsilon)\rho(\mu)\,d\mu = \int_{-\infty}^{\infty} K_2(\lambda - \mu + i\zeta + i\epsilon)\rho(\mu)\,d\mu + \rho(\lambda). \tag{4.38}$$

Combining $\rho(\lambda + i\zeta) = -\rho(\lambda)$ we obtain Eq. (4.33) for $n = 1$. Making analytic continuation with respect to λ we derive Eq. (4.33) for $n = 2, 3, \ldots, 2s - 1$. Similarly, we can show (4.34). □

Proposition 4.2. *When $0 < \zeta < \pi/2s$, matrix elements (A, A) of the spin-s Gaudin matrix with $A = 2s(a-1) + \alpha$ are evaluated by*

$$\frac{1}{2\pi i N_s}\Phi_{A,A}^{(2s)'}\big(\{\lambda_j\}_M\big) = \rho_{\mathrm{tot}}(\mu_a) + O(1/N_s). \tag{4.39}$$

Relations (4.39) are expressed in terms of integrals as follows:

$$\rho_{\mathrm{tot}}(\mu_a) = \frac{1}{N_s}\sum_{p=1}^{N_s} K_{2s}\big(\mu_a - (\alpha - 1/2)\eta - \xi_p + s\eta\big) - \sum_{\gamma=1}^{2s}\int_{-\infty}^{\infty} K_2\big(\mu_a - \mu_c - (\alpha - \gamma)\eta + \epsilon^{(\alpha,\gamma)}\big)\rho_{\mathrm{tot}}(\mu_c)\,d\mu_c, \tag{4.40}$$

where $\epsilon^{(\alpha,\gamma)} = \epsilon_a^{(\alpha)} - \epsilon_c^{(\gamma)}$. We recall that C corresponds to (c, γ) with $C = 2s(c-1) + \gamma$.

Proof. Let us first show

$$K_{2s}\big(\mu_a - (\alpha - 1/2)\eta + s\eta\big) - \sum_{\gamma=1}^{2s}\int_{-\infty}^{\infty} K_2\big(\mu_a - \mu_c + (\gamma - \alpha)\eta + \epsilon^{(\alpha,\gamma)}\big)\rho(\mu_c)\,d\mu_c = \rho(\mu_a) \tag{4.41}$$

for $\alpha = 1, 2, \ldots, 2s$. Making use of the following relations

$$K_n(\lambda) = \frac{1}{2\pi i}\left(\frac{\cosh(\lambda - n\eta/2)}{\sinh(\lambda - n\eta/2)} - \frac{\cosh(\lambda + n\eta/2)}{\sinh(\lambda + n\eta/2)}\right) \tag{4.42}$$

we have

$$K_{2s}\big(\lambda + (2s-1)\eta/2\big) = \sum_{n=0}^{2s-1} K_1(\lambda + n\eta). \tag{4.43}$$

We thus obtain (4.41) as follows:

$$\begin{aligned}
&K_{2s}\big(\mu_a-(\alpha-1/2)\eta+s\eta\big)-\sum_{\gamma=1}^{2s}\int_{-\infty}^{\infty}K_2\big(\mu_a-\mu_c+(\gamma-\alpha)\eta+\epsilon^{(\alpha,\gamma)}\big)\rho(\mu_c)\,d\mu_c\\
&\quad=\sum_{\gamma=1}^{2s}\left(K_1\big(\mu_a+(\gamma-\alpha)\eta\big)-\int_{-\infty}^{\infty}K_2\big(\mu_a-\mu_c+(\gamma-\alpha)\eta+\epsilon^{(\alpha,\gamma)}\big)\rho(\mu_c)\,d\mu_c\right)\\
&\quad=K_1(\mu_a)-\int_{-\infty}^{\infty}K_2(\mu_a-\mu_c)\rho(\mu_c)\,d\mu_c\\
&\quad=\rho(\mu_a).
\end{aligned}\tag{4.44}$$

Here, in the second line of (4.44), the summands for $\gamma\neq\alpha$ vanish due to Lemma 4.1. We then apply the Lieb equation (4.35) to show the last line of (4.44). We obtain (4.40) from (4.41). □

Corollary 4.3. *Let us take a sequence of integers, $c_1,c_2,\ldots,c_{2sm}$, which satisfy $1\leqslant c_j\leqslant M$ for $j=1,2,\ldots,2sm$, and determine a sequence $(z(n))_M$ by the conditions that $\{z(1),z(2),\ldots,z(M')\}=\{1,2,\ldots,M\}\setminus\{c_1,\ldots,c_{2sm}\}$, with $z(1)<\cdots<z(M')$ and $z(j+M')=c_j$ for $j=1,2,\ldots,2sm$. Here we recall $M'=M-2sm$. In the region: $0<\zeta<\pi/2s$, diagonal elements (j,j) of the spin-s Gaudin matrix $\Phi^{(2s)'}((c_k)_{2sm})$ are evaluated as*

$$\frac{1}{2\pi i N_s}\Phi^{(2s)'}_{j,j}\big((c_k)_{2sm}\big)=\rho_{\rm tot}(\mu_a)+O(1/N_s),\quad \textit{for } j=1,2,\ldots,M.\tag{4.45}$$

Here integer a satisfies $z(j)=2s(a-1)+\alpha$ for an integer α with $1\leqslant\alpha\leqslant 2s$.

4.4. Integral equations

We calculate matrix elements of $((\Phi^{(2s)'})^{-1}\Psi^{(2s)'})((c_j)_{2sm})$ through the spin-s Gaudin matrix. For $j,k=1,2,\ldots,M$, we have

$$\begin{aligned}
\big(\Psi^{(2s)'}\big((c_j)_{2sm}\big)\big)_{j,k}&=\big(\Phi^{(2s)'}\big(\Phi^{2s'}\big)^{-1}\Psi^{(2s)'}\big)_{j,k}\\
&=\sum_{t=1}^{M}\big(\Phi^{(2s)'}\big((c_j)_{2sm}\big)\big)_{j,t}\big(\big(\Phi^{(2s)'}\big)^{-1}\Psi^{(2s)'}\big((c_j)_{2sm}\big)\big)_{t,k}.
\end{aligned}\tag{4.46}$$

We remark that matrix elements (A,B) of $\Psi^{(2s)'}((c_l)_{2sm})$ with $A=j+M'$ and $B=k+M'$ are expressed in terms of $K_1(\lambda)$ as

$$\Psi^{(2s)'}_{j+M',k+M'}\big((c_j)_{2sm}\big)/2\pi i=K_1\big(\lambda_{c_j}-w_k^{(2s)}+\eta/2\big)\quad\text{for } j,k=1,2,\ldots,2sm.\tag{4.47}$$

Suppose that we have a sequence $(z(n))_M$ for a given sequence $(c_i)_{2sm}$ satisfying $1\leqslant c_i\leqslant M$ for $i=1,2,\ldots,2sm$. Let us take a pair of integers j,k with $1\leqslant j,k\leqslant M$. We denote $z(j)$ by A, and we introduce a and α by $A=2s(a-1)+\alpha$ with $1\leqslant a\leqslant N_s/2$ and $1\leqslant\alpha\leqslant 2s$. Applying Proposition 4.2 and Corollary 4.3 to (4.46) we have

$$\begin{aligned}
&\sum_{t=1}^{M}\Phi^{(2s)'}_{j,t}\big((c_l)_{2sm}\big)/2\pi i\big(\big(\Phi^{(2s)'}\big)^{-1}\Psi^{(2s)'}\big((c_l)_{2sm}\big)\big)_{t,k}\\
&\quad=\sum_{t=1}^{M}\left\{\left(\sum_{p=1}^{N_s}\frac{1}{2\pi i}\,\frac{\sinh(2s\eta)}{\sinh(\lambda_A-\xi_p)\sinh(\lambda_A-\xi_p+2s\eta)}-\sum_{D=1}^{M}K_2(\lambda_A-\lambda_D)\right)\delta_{A,z(t)}\right.
\end{aligned}$$

$$
\left.+ K_2(\lambda_A - \lambda_{z(t)})\right\}\left(\left(\Phi^{(2s)'}\right)^{-1}\Psi^{(2s)'}\left((c_l)_{2sm}\right)\right)_{t,k}
$$

$$
\begin{aligned}
&= N_s \rho_{\mathrm{tot}}(\mu_a)\left(\left(\Phi^{(2s)'}\right)^{-1}\Psi^{(2s)'}\left((c_j)_{2sm}\right)\right)_{j,k} \\
&\quad + \sum_{t=1}^{M} K_2(\lambda_A - \lambda_{z(t)})\left(\left(\Phi^{(2s)'}\right)^{-1}\Psi^{(2s)'}\left((c_l)_{2sm}\right)\right)_{t,k} + O(1/N_s). \qquad (4.48)
\end{aligned}
$$

Let us discuss the order of magnitude of the correction term in (4.48). It follows from (4.45) that if the density of string centers $\rho(\mu_a)$ is $O(1)$ in the large N_s limit, then the diagonal element (A, A) of $\Phi^{(2s)'}$ is $O(N_s)$. We thus suggest that the matrix elements of $\phi^{(2s;m)}((c_l)_{2sm})$ should be at most of the order of $1/N_s$, and hence the correction term in (4.48) should be at most $O(1/N_s)$.

Let us now define $\varphi_{A,B}(\{\xi_p\})$ by the following relations for $j, k = 1, 2, \ldots, M$:

$$
\varphi_{z(j),k} = N_s \rho_{\mathrm{tot}}(\mu_a)\left(\left(\Phi^{(2s)'}\right)^{-1}\Psi^{(2s)'}\left((c_l)_{2sm}\right)\right)_{j,k}, \qquad (4.49)
$$

where integer a is given by $a = [(z(j) - 1)/2s] + 1$. In terms of function $a(z) = [(z - 1)/2s] + 1$ we have

$$
\begin{aligned}
&\sum_{t=1}^{M} K_2(\lambda_A - \lambda_{z(t)})\left(\left(\Phi^{(2s)'}\right)^{-1}\Psi^{(2s)'}\left((c_l)_{2sm}\right)\right)_{t,k} \\
&\quad = \sum_{t=1}^{M} K_2(\lambda_A - \lambda_{z(t)})\frac{\varphi_{z(t),k}}{N_s \rho_{\mathrm{tot}}(\mu_{a(z(t))})} \\
&\quad = \sum_{\gamma=1}^{2s} \frac{1}{N_s} \sum_{c=1}^{N_s/2} \left(\rho_{\mathrm{tot}}(\mu_c)\right)^{-1} K_2(\lambda_A - \lambda_{(c,\gamma)})\varphi_{2s(c-1)+\gamma,k}. \qquad (4.50)
\end{aligned}
$$

Here we have replaced the sum over t by the sum over c and γ where $z(t) = C = 2s(c - 1) + \gamma$ with $1 \leqslant \gamma \leqslant 2s$. Expressing $z(j)$ and k by A and B, respectively, we have the following equations:

$$
\begin{aligned}
&\varphi_{A,B}\left(\{\xi_p\}\right) + \sum_{\gamma=1}^{2s} \frac{1}{N_s} \sum_{c=1}^{N_s/2} \left(\rho_{\mathrm{tot}}(\mu_c)\right)^{-1} K_2(\lambda_A - \lambda_{(c,\gamma)})\varphi_{C,B}\left(\{\xi_p\}\right) \\
&\quad = \frac{1}{2\pi i}\Psi^{(2s)'}_{A,B}\left((c_l)_{2sm}\right) + O(1/N_s). \qquad (4.51)
\end{aligned}
$$

Let us introduce b and β by $k = 2s(b - 1) + \beta + M'$ with $1 \leqslant \beta \leqslant 2s$ and $1 \leqslant b \leqslant N_s/2$. In terms of string center μ_a we express (or approximate) $\varphi_{z(j),k}(\{\xi_p\})$ by a continuous function of μ_a and ξ_b, as follows:

$$
\varphi_{z(j),k}\left(\{\xi_p\}\right) = \varphi_\alpha^{(\beta)}(\mu_a, \xi_b) + O(1/N_s). \qquad (4.52)
$$

By taking the large-N_s limit, the discrete equations (4.51) are now expressed as follows:

$$
\begin{aligned}
&\varphi_\alpha^{(\beta)}(\mu_a, \xi_b) + \sum_{\gamma=1}^{2s} \int_{-\infty}^{\infty} K_2\left(\mu_a - \mu_c + (\gamma - \alpha)\eta + \epsilon_{AC}\right)\varphi_\gamma^{(\beta)}(\mu_c, \xi_b)\, d\mu_c \\
&\quad = K_1\left(\lambda_{c_{j-M'}} - w_k^{(2s)} + \eta/2\right). \qquad (4.53)
\end{aligned}
$$

Here $\epsilon_{AC} = \epsilon_a^{(\alpha)} - \epsilon_c^{(\gamma)}$. We recall that for $j > M'$ we have set $z(j) = c_{j-M'}$.

Lemma 4.4. *In the region* $0 < \zeta < \pi/2s$, *a solution to the integral equations* (4.53) *for integers* $A = 2s(a-1)+\alpha$ *(i.e.* (a,α)*) and* $B = 2s(b-1)+\beta+M'$ *with* $1 \leqslant \alpha, \beta \leqslant 2s$ *and* $1 \leqslant b \leqslant m$ *is given by*

$$\varphi_{A,B} = \varphi_\alpha^{(\beta)}(\mu_a, \xi_b) = \rho(\mu_a - \xi_b)\,\delta_{\alpha,\beta}. \tag{4.54}$$

Proof. (i) In (α,α) case, i.e. when integers $A = 2s(a-1)+\alpha$ and $B = 2s(b-1)+\alpha$ correspond to indices (a,α) and (b,α), respectively, assuming that $\varphi_\gamma^{(\alpha)}(\mu_c,\xi_b) = 0$ for $\gamma \neq \alpha$, we reduce integral equations (4.53) to the Lieb equation (4.35). Therefore, we have $\varphi_\alpha^{(\alpha)}(\mu_a,\xi_b) = \rho(\mu_a - \xi_b)$.

(ii) In (α,β) case, i.e. when $A = (a,\alpha)$ and $B = (b,\beta)$ with $\beta \neq \alpha$, assuming $\varphi_\gamma^{(\beta)}(\mu_c,\xi_b) = 0$ for $\gamma \neq \beta$, we have from (4.53)

$$\int_{-\infty}^{\infty} K_2\big(\mu_a - \mu_c + (\beta-\alpha)\eta + \epsilon_{AB}\big)\varphi_\beta^{(\beta)}(\mu_c,\xi_b)\,d\mu_c = \frac{1}{2\pi i}\,\Psi_{A,B}^{(2s)}. \tag{4.55}$$

For $\alpha < \beta$, we have $\epsilon_{AB} = i\epsilon$. Shifting μ_a analytically such as $\mu_a \to \mu_a - (\beta-\alpha-1)\eta$, we have

$$\int_{-\infty}^{\infty} K_2(\mu_a - \mu_c + \eta + i\epsilon)\varphi_\beta^{(\beta)}(\mu_c,\xi_b)\,d\mu_c$$
$$= \frac{1}{2\pi i}\,\frac{\sinh\eta}{\sinh(\mu_a+\eta-\xi_b-\eta/2)\sinh(\mu_a+\eta-\xi_b+\eta/2)}. \tag{4.56}$$

Making use of (4.37) we reduce it essentially to the Lieb equation. We thus obtain $\varphi_\beta^{(\beta)}(\mu_a,\xi_b) = \rho(\mu_a - \xi_b)$. For $\alpha > \beta$, we have $\epsilon_{AB} = -i\epsilon$, and show it similarly, shifting μ_a analytically as $\mu_a \to \mu_a - (\alpha-\beta+1)\eta$. □

Proposition 4.5. *Let us take a set of integers,* $c_1,\ldots,c_{2sm}$, *satisfying* $0 < c_j \leqslant M$ *for* $j = 1,2,\ldots,2sm$. *Suppose that the number of* c_j *which satisfy* $c_j - 2s[(c_j-1)/2s] = \alpha$ *is given by* m *for each integer* α *satisfying* $1 \leqslant \alpha \leqslant 2s$. *Then, when* $0 < \zeta < \pi/2s$, *the solution to integral equations* (4.53) *for* $A = c_j$ *with* $j = 1,2,\ldots,2sm$ *and for* $B = B' + M'$ *where* $B' = 1,2,\ldots,2sm$, *is given by*

$$\varphi_\alpha^{(\beta)}(\mu_{a_j},\xi_b) = \rho(\mu_{a_j} - \xi_b)\delta_{\alpha,\beta}, \tag{4.57}$$

where $a_j = [(c_j-1)/2s]+1$, $\alpha = c_j - 2s[(c_j-1)/2s]$ *and* $B' = 2s(b-1)+\beta$ *with* $1 \leqslant \beta \leqslant 2s$.

Proof. It follows from Lemma 4.4 that $\varphi_{c_j,B}$ of (4.57) gives a solution to the integral equations. Taking the Fourier transform of (4.53), we show in Section 4.5 that the set of integral equations (4.53) for $A = c_j$ for $j = 1,2,\ldots,2sm$ and $B' = 1,2,\ldots,2sm$ has a unique solution. Thus we obtain the unique solution (4.57). □

Let us recall the assumption that function $\varphi_\alpha^{(\beta)}(\mu_a,\xi_b)$ is continuous with respect to μ_a and ξ_b. Then, for any given set of integers, $c_1,\ldots,c_{2sm}$ satisfying $0 < c_j \leqslant M$ for $j = 1,2,\ldots,2sm$, we may approximate the matrix elements of $(\Phi^{(2s)'})^{-1}\Psi^{(2s)'}$ as follows. For integers j and k with $1 \leqslant j,k \leqslant 2sm$, we define a_j, α_j, b_k and β_k as follows:

$$
\begin{aligned}
a_j &= \big[(c_j-1)/2s\big]+1, \qquad \alpha_j = c_j - 2s\big[(c_j-1)/2s\big], \\
b_k &= \big[(k-1)/2s\big]+1, \qquad \beta_k = k - 2s\big[(k-1)/2s\big].
\end{aligned} \tag{4.58}
$$

Then, we have

$$
\big(\big(\Phi^{(2s)'}\big)^{-1}\Psi^{(2s)'}\big((c_j)_{2sm}\big)\big)_{j+M',k+M'} = \frac{1}{N_s}\frac{\rho(\mu_{a_j}-\xi_{b_k})}{\rho_{\mathrm{tot}}(\mu_{a_j})}\delta_{\alpha_j,\beta_k} + O\big(1/N_s^2\big). \tag{4.59}
$$

Here we recall $M' = M - 2sm$.

For a given $2s$-string, $\lambda_{(a,\alpha)}$, with $\alpha = 1, 2, \ldots, 2s$, we define $\lambda'_{(a,\alpha)}$ by the 'regular part' of $\lambda_{(a,\alpha)}$:

$$
\lambda'_{(a,\alpha)} = \mu_a - (\alpha - 1/2)\eta. \tag{4.60}
$$

Let us introduce a $2sm$-by-$2sm$ matrix S by

$$
\begin{aligned}
&S_{j,k}\big(c_1, \ldots, c_{2sm}; (\xi_p)_{N_s}\big) = \rho\big(\lambda'_{c_j} - w_k^{(2s)} + \eta/2\big)\delta_{\alpha_j,\beta_k} \\
&\quad \text{for } j, k = 1, 2, \ldots, 2sm.
\end{aligned} \tag{4.61}
$$

Here a_j, α_j, b_k and β_k are given by (4.58). Then, we obtain

$$
\phi_{j,k}^{(2s;m)}\big((c_k)_{2sm}; \{\xi_p\}\big) = \frac{1}{N_s}\frac{1}{\rho_{\mathrm{tot}}(\mu_a)} S_{j,k}\big((c_k)_{2sm}; (\xi_p)_{N_s}\big) + O\big(1/N_s^2\big) \tag{4.62}
$$

and we have

$$
\begin{aligned}
&\det\big(\phi^{(2s;m)}\big((c_k)_{2sm}; \{\xi_p\}\big)\big) \\
&\quad = \prod_{j=1}^{2sm}\left(\frac{1}{N_s}\frac{1}{\rho_{\mathrm{tot}}(\mu_{a_j})}\right)\cdot\big(\det S\big((c_j)_{2sm}; (\xi_p)_{N_s}\big) + O(1/N_s)\big).
\end{aligned} \tag{4.63}
$$

4.5. Fourier transform in the cases of spin-1 and general spin-s

The integral equations (4.53) for the spin-1 case are given by

$$
\begin{cases}
\varphi_1^{(1)}(\mu,\xi_p) + \int_{-\infty}^{\infty} K_2(\mu-\lambda)\varphi_1^{(1)}(\lambda,\xi_p)\,d\lambda \\
\qquad + \int_{-\infty}^{\infty} K_2(\mu-\lambda+\eta+\epsilon^{(1,2)})\varphi_2^{(1)}(\lambda,\xi_p)\,d\lambda \\
\quad = \frac{1}{2\pi i}\frac{\sinh(\eta)}{\sinh(\mu-\xi_p+\frac{\eta}{2})\sinh(\mu-\xi_p-\frac{\eta}{2})} \\
\varphi_1^{(2)}(\mu,\xi_p) + \int_{-\infty}^{\infty} K_2(\mu-\lambda)\varphi_1^{(2)}(\lambda,\xi_p)\,d\lambda \\
\qquad + \int_{-\infty}^{\infty} K_2(\mu-\lambda+\eta+\epsilon^{(1,2)})\varphi_2^{(2)}(\lambda,\xi_p)\,d\lambda \\
\quad = \frac{1}{2\pi i}\frac{\sinh(\eta)}{\sinh(\mu-\xi_p+\frac{\eta}{2})\sinh(\mu-\xi_p+\frac{3\eta}{2})} \\
\varphi_2^{(1)}(\mu,\xi_p) + \int_{-\infty}^{\infty} K_2(\mu-\lambda-\eta+\epsilon^{(2,1)})\varphi_1^{(1)}(\lambda,\xi_p)\,d\lambda \\
\qquad + \int_{-\infty}^{\infty} K_2(\mu-\lambda)\varphi_2^{(1)}(\lambda,\xi_p)\,d\lambda \\
\quad = \frac{1}{2\pi i}\frac{\sinh(\eta)}{\sinh(\mu-\xi_p-\frac{3\eta}{2})\sinh(\mu-\xi_p-\frac{\eta}{2})} \\
\varphi_2^{(2)}(\mu,\xi_p) + \int_{-\infty}^{\infty} K_2(\mu-\lambda-\eta+\epsilon^{(2,1)})\varphi_1^{(2)}(\lambda,\xi_p)\,d\lambda \\
\qquad + \int_{-\infty}^{\infty} K_2(\mu-\lambda)\varphi_2^{(2)}(\lambda,\xi_p)\,d\lambda \\
\quad = \frac{1}{2\pi i}\frac{\sinh(\eta)}{\sinh(\mu-\xi_p-\frac{\eta}{2})\sinh(\mu-\xi_p+\frac{\eta}{2})}.
\end{cases} \tag{4.64}
$$

We solve integral equations (4.53) via the Fourier transform. Let us express the Fourier transform of function $\varphi_\alpha^{(\beta)}(\mu,\xi)$ by

$$\widehat{\varphi}_\alpha^{(\beta)}(\omega,\xi) = \int_{-\infty}^{\infty} e^{i\mu\omega}\varphi_\alpha^{(\beta)}(\mu,\xi)\,d\mu, \quad \text{for } \alpha,\beta = 1,2,\dots,2s. \tag{4.65}$$

We denote by $\widehat{K}_n(\omega)$ the Fourier transform of kernel $K_n(\lambda)$. We define matrix $\mathcal{M}_{\hat{\varphi}}^{(2s)}$ by

$$\big(\mathcal{M}_{\hat{\varphi}}^{(2s)}\big)_{\alpha\beta} = \widehat{\varphi}_\alpha^{(\beta)}(\omega,\xi) \quad \text{for } \alpha,\beta = 1,2,\dots,2s. \tag{4.66}$$

We introduce a $2s$-by-$2s$ matrix $\mathcal{M}_{K_2}^{(2s)}$. We define matrix element (j,k) for $j,k = 1,2,\dots,2sm$, by

$$\big(\mathcal{M}_{K_2}^{(2s)}\big)_{j,k} = \begin{cases} 1 + \int_{-\infty}^{\infty} e^{i\mu\omega} K_2(\mu)\,d\mu & \text{for } j = k, \\ \int_{-\infty}^{\infty} e^{i\mu\omega} K_2(\mu + (k-j)\eta + i0)\,d\mu & \text{for } j < k, \\ \int_{-\infty}^{\infty} e^{i\mu\omega} K_2(\mu - (j-k)\eta - i0)\,d\mu & \text{for } j > k. \end{cases} \tag{4.67}$$

When $0 < \zeta < \pi/2s$, we calculate the matrix elements of $\mathcal{M}_{K_2}^{(2s)}$ as follows:

$$\big(\mathcal{M}_{K_2}^{(2s)}\big)_{j,k} = \delta_{j,k}\big(1 + \widehat{K}_2(\omega)\big) + (1-\delta_{j,k})e^{(k-j)\zeta\omega}\big(\widehat{K}_2(\omega) - e^{\mathrm{sgn}(j-k)\zeta\omega}\big)$$
$$\text{for } j,k = 1,2,\dots,2s. \tag{4.68}$$

Here we define $\mathrm{sgn}(j-k)$ by the following: $\mathrm{sgn}(j-k) = -1$ for $j-k<0$, and $\mathrm{sgn}(j-k) = +1$ for $j-k>0$. Here, $\widehat{K}_2(\omega)$, is given by

$$\widehat{K}_2(\omega) = \int_{-\infty}^{\infty} e^{i\omega\mu} K_2(\mu)\,d\mu = \frac{\sinh(\frac{\pi}{2}-\zeta)\omega}{\sinh(\frac{\pi\omega}{2})}. \tag{4.69}$$

Similarly, we define a $2s$-by-$2s$ matrix $\mathcal{M}_{K_1}^{(2s)}$ by

$$\big(\mathcal{M}_{K_1}^{(2s)}\big)_{j,k} = \int_{-\infty}^{\infty} e^{i\omega\mu} K_1\big(\mu - \xi_b + (k-j)\eta\big)\,d\mu \quad \text{for } j,k = 1,2,\dots,2s. \tag{4.70}$$

When $0 < \zeta < \pi/2s$, we can show

$$\big(\mathcal{M}_{K_1}^{(2s)}\big)_{j,k} = e^{i\xi_b\omega}\big\{\delta_{j,k}\widehat{K}_1(\omega) + (1-\delta_{j,k})\,e^{(k-j)\zeta\omega}\big(\widehat{K}_1(\omega) - e^{\mathrm{sgn}(j-k)\zeta\omega/2}\big)\big\} \tag{4.71}$$

for $j,k = 1,2,\dots,2s$. Here $\widehat{K}_1(\omega)$ is given by

$$\widehat{K}_1(\omega) = \int_{-\infty}^{\infty} e^{i\omega\mu} K_1(\mu)\,d\mu = \frac{\sinh(\frac{\pi}{2}-\frac{\zeta}{2})\omega}{\sinh(\frac{\pi\omega}{2})}. \tag{4.72}$$

Taking the Fourier transform of integral equations (4.53) we have the following matrix equation

$$\mathcal{M}_{K_2}^{(2s)}\mathcal{M}_{\hat{\varphi}}^{(2s)} = \mathcal{M}_{K_1}^{(2s)}. \tag{4.73}$$

For the spin-1 case, from (4.64) we have

$$\begin{pmatrix} 1+\widehat{K}_2(\omega) & e^{\zeta\omega}\widehat{K}_2(\omega)-1 \\ e^{-\zeta\omega}\widehat{K}_2(\omega)-1 & 1+\widehat{K}_2(\omega) \end{pmatrix} \begin{pmatrix} \widehat{\varphi}_1^{(1)}(\omega) & \widehat{\varphi}_1^{(2)}(\omega) \\ \widehat{\varphi}_2^{(1)}(\omega) & \widehat{\varphi}_2^{(2)}(\omega) \end{pmatrix}$$
$$= e^{i\xi_b\omega} \begin{pmatrix} \widehat{K}_1(\omega) & e^{\zeta\omega}\widehat{K}_1(\omega)-e^{\zeta\omega/2} \\ e^{-\zeta\omega}\widehat{K}_1(\omega)-e^{-\zeta\omega/2} & \widehat{K}_1(\omega) \end{pmatrix}. \tag{4.74}$$

It is easy to show that matrix $M^{(2)}(\widehat{\varphi})$ is given by the following:

$$\begin{pmatrix} \widehat{\varphi}_1^{(1)}(\omega) & \widehat{\varphi}_1^{(2)}(\omega) \\ \widehat{\varphi}_2^{(1)}(\omega) & \widehat{\varphi}_2^{(2)}(\omega) \end{pmatrix} = \begin{pmatrix} \frac{e^{i\xi_b\omega}}{2\cosh(\zeta\omega/2)} & 0 \\ 0 & \frac{e^{i\xi_b\omega}}{2\cosh(\zeta\omega/2)} \end{pmatrix}. \tag{4.75}$$

We calculate the determinant of $\mathcal{M}_{K_2}^{(2)}$ (the spin-1 case) as follows:

$$\begin{aligned} \det\begin{pmatrix} 1+\widehat{K}_2(\omega) & e^{\zeta\omega}\widehat{K}_2(\omega)-1 \\ e^{-\zeta\omega}\widehat{K}_2(\omega)-1 & 1+\widehat{K}_2(\omega) \end{pmatrix} &= \det\begin{pmatrix} 1+\widehat{K}_2(\omega) & e^{\zeta\omega}\widehat{K}_2(\omega)-1 \\ -(1+e^{-\zeta\omega}) & 1+e^{-\zeta\omega} \end{pmatrix} \\ &= \det\begin{pmatrix} \widehat{K}_2(\omega)(1+e^{\zeta\omega}) & e^{\zeta\omega}\widehat{K}_2(\omega)-1 \\ 0 & 1+e^{-\zeta\omega} \end{pmatrix} \\ &= \widehat{K}_2(\omega)\bigl(1+e^{-\zeta\omega}\bigr)\bigl(1+e^{\zeta\omega}\bigr). \end{aligned} \tag{4.76}$$

Here, we first subtract the 2nd row by the 1st row multiplied by $e^{-\zeta\omega}$. We next add the 2nd column to the 1st column. Finally, the determinant is factorized and we have the result.

By the same method we can calculate the determinant of $\mathcal{M}_{K_2}^{(2s)}$ for the spin-s case. We have

$$\det\mathcal{M}_{K_2}^{(2s)} = \bigl(2\cosh(\zeta\omega/2)\bigr)^{2s}\,\widehat{K}_{2s}(\omega). \tag{4.77}$$

The determinant is nonzero generically, and hence the solution to matrix equation (4.73) is unique. Therefore, we obtain the solution of integral equation (4.53).

5. The EFP of the spin-s XXZ spin chain near AF point

5.1. *Multiple-integral representations of the spin-s EFP*

Let us derive multiple-integral representations for the emptiness formation probability of the spin-s XXZ spin chain. We shall take the large N_s limit of the EFP (4.30) for a finite-size system, and we replace rapidity λ_{c_j} with complex variable λ_j for $j=1,2,\ldots,2sm$, as follows. For a given rapidity of $2s$-string, $\lambda_A=\mu_a-(\alpha-1/2)\eta+\epsilon_A$, we define its regular part λ'_A by $\lambda'_A=\mu_a-(\alpha-1/2)\eta$. In the large-$N_s$ limit, we first replace $\lambda_{c_k}=\lambda'_{c_k}+\epsilon_{c_k}$ by $\lambda'_k+\epsilon_{c_k}$ where λ'_k are complex integral variables corresponding to complete strings such as $\lambda'_k=\mu_k-(\beta-1/2)\eta$ for some integer β with $1\leqslant\beta\leqslant 2s$ where $\eta=i\zeta$ with $0<\zeta<\pi$ and μ_k is real. We express λ'_k and ϵ_{c_k} simply by λ_k and ϵ_k, respectively, and then we obtain multiple-integral representations.

Applying (4.63) we derive the emptiness formation probability for arbitrary spin-s in the thermodynamic limit $N_s\to\infty$, as follows:

$$
\begin{aligned}
&\tau_{\infty}^{(2s)}\bigl(m;\{\xi_p\}\bigr)\\
&\quad=\frac{1}{\prod_{1\leqslant j<r\leqslant 2s}(\sinh(r-j)\eta)^m}\times\frac{1}{\prod_{1\leqslant k<l\leqslant m}\prod_{j=1}^{2s}\prod_{r=1}^{2s}\sinh(\xi_k-\xi_l+(r-j)\eta)}\\
&\qquad\times\prod_{l=1}^{2sm}\left(\sum_{k=1}^{2s}\int\limits_{-\infty+(-k+\frac{1}{2})\eta}^{\infty+(-k+\frac{1}{2})\eta}d\lambda_l\right)H^{(2s)}(\lambda_1,\ldots,\lambda_{2sm})\det S(\lambda_1,\ldots,\lambda_{2sm})
\end{aligned}
\tag{5.1}
$$

where $H^{(2s)}((\lambda_l)_{2sm})$ is given by

$$
\begin{aligned}
&H^{(2s)}\bigl((\lambda_l)_{2sm}\bigr)\\
&\quad=\frac{1}{\prod_{1\leqslant l<k\leqslant 2sm}\sinh(\lambda_k-\lambda_l+\eta+\epsilon_{k,l})}\times\prod_{l=1}^{2sm}\prod_{k=1}^{m}\prod_{p=1}^{2s-1}\sinh\bigl(\lambda_l-\xi_k+(2s-p)\eta\bigr)\\
&\qquad\times\prod_{l=1}^{m}\prod_{r_l=1}^{2s}\left(\prod_{k=1}^{l-1}\sinh(\lambda_{2s(l-1)+r_l}-\xi_k+2s\eta)\prod_{k=l+1}^{m}\sinh(\lambda_{2s(l-1)+r_l}-\xi_k)\right)
\end{aligned}
\tag{5.2}
$$

and matrix elements of $S(\lambda_1,\ldots,\lambda_{2sm})$ are given by

$$
S_{j,2s(l-1)+k}=\begin{cases}\rho(\lambda_j-\xi_l+(k-\frac{1}{2})\eta) & \text{if } \lambda_j-\mu_j=(\frac{1}{2}-k)\eta,\\ 0 & \text{otherwise.}\end{cases}
\tag{5.3}
$$

Here μ_j denotes the center of the $2s$-string in which λ_j is the kth rapidity. Explicitly, we have $\lambda_j=\mu_j-(k-1/2)\eta$. In the denominator, we have set $\epsilon_{k,l}$ associated with λ_k and λ_l as follows:

$$
\epsilon_{k,l}=\begin{cases} i\epsilon & \text{for } \mathcal{I}m(\lambda_k-\lambda_l)>0,\\ -i\epsilon & \text{for } \mathcal{I}m(\lambda_k-\lambda_l)<0.\end{cases}
\tag{5.4}
$$

In the homogeneous case we have $\epsilon_p=0$ for $p=1,2,\ldots,N_s$. We have thus defined inhomogeneous parameters ξ_p. We recall that in the homogeneous case, the spin-s Hamiltonian is derived from the logarithmic derivative of the spin-s transfer matrix.

Here we should remark that an expression of the matrix elements of S similar to (5.3) has been given in Eq. (6.14) of Ref. [13] for the correlation functions of the integrable spin-s XXX spin chain.

5.2. Symmetric expression of the spin-s EFP

We shall express the spin-s EFP (5.1) in a simpler way, making use of permutations of $2sm$ integers, $1,2,\ldots,2sm$, and the formula of the Cauchy determinant.

Let us take a set of integers $a_{(j,k)}=a_{2s(j-1)+k}$ satisfying $1\leqslant a_{(j,k)}\leqslant N_s/2$ for $j=1,2,\ldots,m$ and $k=1,2,\ldots,2s$. Here we remark that indices $a_{(j,k)}$ correspond to the string centers $\mu_{(j,k)}=\mu_{2s(j-1)+k}$. In order to reformulate the sum over integers, $c_1,\ldots,c_{2sm}$, in Eq. (4.30) in terms of indices $a_{(j,k)}$, let us introduce $\hat{c}_1,\ldots,\hat{c}_{2sm}$ by

$$
\hat{c}_{2s(j-1)+k}=2s(a_{(j,k)}-1)+k,\quad \text{for } j=1,2,\ldots,m \text{ and } k=1,2,\ldots,2s.
\tag{5.5}
$$

We also define $\beta(z)$ by $\beta(z)=z-2s[(z-1)/2s]$. Then, $\hat{c}_j$ are expressed as follows:

$$
\hat{c}_j=2s(a_j-1)+\beta(j),\quad \text{for } j=1,2,\ldots,2sm.
\tag{5.6}
$$

We decompose the sum over c_j into $2s$ sums over a_j as follows:

$$\sum_{c_j=1}^{M} g(c_j) = \sum_{k=1}^{2s} \sum_{a_j=1}^{N_s/2} g\bigl(2s(a_j - 1) + k\bigr). \tag{5.7}$$

Let us consider such a function $f(c_1, c_2, \ldots, c_{2sm})$ of sequence of integers $(c_j)_{2sm}$ that vanishes unless c_j's are distinct. We also assume that $f(c_1, c_2, \ldots, c_{2sm})$ vanishes unless the number of c_j's satisfying $\beta(c_j) = \alpha$ is given by m for each integer α satisfying $1 \leqslant \alpha \leqslant 2s$. Here we recall that the two properties are in common with the summand of (4.30), in particular, with $\det(\phi^{(2s;m)}(c_j)_{2sm}; \{\xi_p\})$. Then, we have

$$\begin{aligned}
&\sum_{c_1=1}^{M} \sum_{c_2=1}^{M} \cdots \sum_{c_{2sm}=1}^{M} f(c_1, \ldots, c_{2sm}) \\
&\quad = \frac{1}{(m!)^{2s}} \sum_{a_1=1}^{N_s/2} \sum_{a_2=1}^{N_s/2} \cdots \sum_{a_{2sm}=1}^{N_s/2} \sum_{P \in \mathcal{S}_{2sm}} f(\hat{c}_{P1}, \ldots, \hat{c}_{P(2sm)}) \\
&\quad = \sum_{a_1=1}^{N_s/2} \sum_{a_2=1}^{N_s/2} \cdots \sum_{a_{2sm}=1}^{N_s/2} \sum_{\pi \in \mathcal{S}_{2sm}/(\mathcal{S}_m)^{2s}} f(\hat{c}_{\pi 1}, \ldots, \hat{c}_{\pi(2sm)}).
\end{aligned} \tag{5.8}$$

Here an element π of $\mathcal{S}_{2sm}/(\mathcal{S}_m)^{2s}$ gives a permutation of integers $1, 2, \ldots, 2sm$, where πj's such that $\pi j \equiv k \pmod{2s}$ are put in increasing order in the sequence $(\pi 1, \pi 2, \ldots, \pi(2sm))$ for $k = 0, 1, \ldots, 2s - 1$.

Reformulating the sum over c_j s in (4.30) in terms of a_j's, in the large N_s limit we have

$$\begin{aligned}
&\tau_\infty^{(2s)}(m) \\
&\quad = \frac{1}{\prod_{1 \leqslant \alpha < \beta \leqslant 2s} (\sinh(\beta - \alpha)\eta)^m} \times \frac{1}{\prod_{1 \leqslant k < l \leqslant m} \prod_{\alpha=1}^{2s} \prod_{\beta=1}^{2s} \sinh(\xi_k - \xi_l + (\alpha - \beta)\eta)} \\
&\qquad \times \prod_{k=1}^{2s} \prod_{j=1}^{m} \int_{-\infty}^{\infty} d\mu_{(j,k)} \sum_{P \in \mathcal{S}_{2sm}} \frac{1}{(m!)^{2s}} \det S(\lambda_{P1}, \ldots, \lambda_{P(2sm)}) H^{(2s)}(\lambda_{P1}, \ldots, \lambda_{P(2sm)}) \\
&\quad = \frac{1}{\prod_{1 \leqslant \alpha < \beta \leqslant 2s} (\sinh(\beta - \alpha)\eta)^m} \times \frac{1}{\prod_{1 \leqslant k < l \leqslant m} \prod_{\alpha=1}^{2s} \prod_{\beta=1}^{2s} \sinh(\xi_k - \xi_l + (\alpha - \beta)\eta)} \\
&\qquad \times \prod_{j=1}^{2sm} \int_{-\infty}^{\infty} d\mu_j \sum_{\pi \in \mathcal{S}_{2sm}/(\mathcal{S}_m)^{2s}} \det S(\lambda_{\pi 1}, \ldots, \lambda_{\pi(2sm)}) H^{(2s)}(\lambda_{\pi 1}, \ldots, \lambda_{\pi(2sm)}),
\end{aligned} \tag{5.9}$$

where symbols λ_j denote the following

$$\lambda_j = \mu_j - \left(\beta(j) - \frac{1}{2}\right)\eta \quad \text{for } j = 1, 2, \ldots, 2sm. \tag{5.10}$$

We calculate $\det S$ applying the Cauchy determinant formula

$$\det\left(\frac{1}{\sinh(\lambda_a - \xi_k)}\right) = \frac{\prod_{k<l}^{m} \sinh(\xi_k - \xi_l) \prod_{a>b}^{m} \sinh(\lambda_a - \lambda_b)}{\prod_{a=1}^{m} \prod_{k=1}^{m} \sinh(\lambda_a - \xi_k)}, \tag{5.11}$$

and we obtain the symmetric expression of the spin-s EFP as follows:

$$\begin{aligned}
&\tau_\infty^{(2s)}\big(m;\{\xi_p\}\big)\\
&= \frac{1}{\prod_{1\leqslant\alpha<\beta\leqslant 2s}\sinh^m(\beta-\alpha)\eta}\prod_{1\leqslant k<l\leqslant m}\frac{\sinh^{2s}(\pi(\xi_k-\xi_l)/\zeta)}{\prod_{j=1}^{2s}\prod_{r=1}^{2s}\sinh(\xi_k-\xi_l+(r-j)\eta)}\\
&\quad\times\frac{i^{2sm^2}}{(2i\zeta)^{2sm}}\left(\prod_{j=1}^{2sm}\int_{-\infty}^{\infty}d\mu_j\right)\prod_{\gamma=1}^{2s}\prod_{1\leqslant b<a\leqslant m}\sinh\big(\pi(\mu_{2s(a-1)+\gamma}-\mu_{2s(b-1)+\gamma})/\zeta\big)\\
&\quad\times\left(\prod_{j=1}^{2sm}\frac{\prod_{b=1}^{m}\prod_{\beta=1}^{2s-1}\sinh(\lambda_j-\xi_b+\beta\eta)}{\prod_{b=1}^{m}\cosh(\pi(\mu_j-\xi_b)/\zeta)}\right)\sum_{\sigma\in\mathcal{S}_{2sm}/(\mathcal{S}_m)^{2s}}(\operatorname{sgn}\sigma)\\
&\quad\times\left(\prod_{l=1}^{m}\prod_{r_l=1}^{2s}\left(\prod_{k=1}^{l-1}\sinh(\lambda_{\sigma(2s(l-1)+r_l)}-\xi_k+2s\eta)\prod_{k=l+1}^{m}\sinh(\lambda_{\sigma(2s(l-1)+r_l)}-\xi_k)\right)\right)\\
&\quad\times\left(\prod_{1\leqslant l<k\leqslant 2sm}\sinh(\lambda_{\sigma(k)}-\lambda_{\sigma(l)}+\eta+\epsilon_{\sigma(k),\sigma(l)})\right)^{-1}.
\end{aligned}\tag{5.12}$$

Here $(\operatorname{sgn}\sigma)$ denotes the sign of permutation $\sigma\in\mathcal{S}_{2sm}/(\mathcal{S}_m)^{2s}$.

5.3. The spin-s EFP for the homogeneous chain

Sending ξ_b to zero for $b=1,2,\ldots,m$, we derive the spin-s EFP in the homogeneous limit. Here we remark that the spin-s Hamiltonian is derived from the logarithmic derivative of the spin-s transfer matrix $t^{(2s,2s)}(\lambda;\{\xi_b\}_{N_s})$ in the homogeneous case where $\xi_b=0$ for $b=1,2,\ldots,N_s$.

Sending ξ_b to zero for $b=1,2,\ldots,m$, we have the following:

$$\begin{aligned}
&\lim_{\xi_1\to 0}\lim_{\xi_2\to 0}\cdots\lim_{\xi_m\to 0}\big(\tau_\infty^{(2s)}\big(m;\{\xi_p\}_m\big)\big)\\
&=\frac{(\pi/\zeta)^{sm(m-1)}}{\prod_{1\leqslant\alpha<\beta\leqslant 2s}\sinh^{m^2}((\beta-\alpha)\eta)}\\
&\quad\times\frac{i^{2sm^2}}{(2i\zeta)^{2sm}}\prod_{j=1}^{2sm}\int_{-\infty}^{\infty}d\mu_j\prod_{\beta=1}^{2s}\prod_{1\leqslant b<a\leqslant m}\sinh\big(\pi(\mu_{2s(a-1)+\beta}-\mu_{2s(b-1)+\beta})/\zeta\big)\\
&\quad\times\left(\prod_{j=1}^{2sm}\frac{\prod_{\beta=1}^{2s-1}\sinh^m(\lambda_j+\beta\eta)}{\cosh^m(\pi\mu_j/\zeta)}\right)\times\sum_{\sigma\in\mathcal{S}_{2sm}/(\mathcal{S}_m)^{2s}}(\operatorname{sgn}\sigma)\\
&\quad\times\frac{\prod_{l=1}^{m}\prod_{r_l=1}^{2s}(\sinh^{l-1}(\lambda_{\sigma(2s(l-1)+r_l)}+2s\eta)\sinh^{m-l}(\lambda_{\sigma(2s(l-1)+r_l)}))}{\prod_{1\leqslant l<k\leqslant 2sm}\sinh(\lambda_{\sigma(k)}-\lambda_{\sigma(l)}+\eta+\epsilon_{\sigma(k),\sigma(l)})}.
\end{aligned}\tag{5.13}$$

Here we recall definition (5.10) of λ_j.

Let us discuss that expression (5.13) gives the spin-s EFP for the homogeneous chain. First, we remark that $\tau_{N_s}^{(2s)}(m;\{\xi_p\}_m)$ does not depend on ξ_p with $p>m$. Hence we may consider that inhomogeneous parameters ξ_p with $p > m$ are all set to be zero, after computing the EFP: $\tau_{N_s}^{(2s)}(m; \{\xi_p\}_m)$.

We now show that the order of the homogeneous limit: $\xi_p \to 0$ and the thermodynamic limit $N_s \to \infty$ can be reversed. We can show the following relation:

$$\prod_{p=1}^{m} \lim_{\xi_p \to 0}\Bigl(\lim_{N_s\to\infty}\bigl(\tau_{N_s}^{(2s)}\bigl(m; \{\xi_p\}_m\bigr)\bigr)\Bigr) = \lim_{N_s\to\infty}\left(\prod_{p=1}^{m} \lim_{\xi_p\to 0}\bigl(\tau_{N_s}^{(2s)}\bigl(m; \{\xi_p\}_m\bigr)\bigr)\right). \tag{5.14}$$

In fact, when N_s is very large, it follows from (4.63) that we have

$$\tau_{\infty}^{(2s)}\bigl(m; \{\xi_p\}_m\bigr) = \tau_{N_s}^{(2s)}\bigl(m; \{\xi_p\}_m\bigr) + O(1/N_s). \tag{5.15}$$

Furthermore, we can explicitly show that $\tau_{N_s}^{(2s)}(m; \{\xi_p\}_m)$ is continuous with respect to ξ_p at $\xi_p = 0$ for $p = 1, 2, \ldots, m$. We first reformulate the sum over c_j in (4.30) into the sum over $a_{(j,k)}$ by relation (5.8)

$$\begin{aligned}
&\sum_{c_1=1}^{M}\cdots\sum_{c_{2sm}=1}^{M} \det S\bigl((c_j)_{2sm}\bigr) H^{(2s)}\bigl((\lambda_{c_j})_{2sm}\bigr) \\
&\quad = \prod_{j=1}^{m}\prod_{k=1}^{2s}\left(\frac{1}{m!}\sum_{a_{(j,k)}=1}^{N_s/2}\right) \sum_{P\in\mathcal{S}_{2sm}} \det S\bigl((\hat{c}_{Pj})_{2sm}\bigr) H^{(2s)}\bigl((\lambda_{\hat{c}_j})_{2sm}\bigr) \\
&\quad = \prod_{j=1}^{m}\prod_{k=1}^{2s}\left(\frac{1}{m!}\sum_{a_{(j,k)}=1}^{N_s/2}\right) \det S\bigl((\hat{c}_j)_{2sm}\bigr) \sum_{P\in\mathcal{S}_{2sm}} (\operatorname{sgn} P) H^{(2s)}\bigl((\lambda_{\hat{c}_j})_{2sm}\bigr).
\end{aligned} \tag{5.16}$$

We then apply the Cauchy determinant formula to evaluate $\det S(\hat{c}_j)$ as follows:

$$\begin{aligned}
&\det S\bigl((\hat{c}_j)_{2sm}\bigr) \\
&\quad = \det S\bigl((\lambda_{\hat{c}_j})_{2sm}\bigr) = \prod_{\alpha=1}^{2s} \det S^{(\alpha)}\bigl((\lambda_{2s(a_{(j,\alpha)}-1)+\alpha})_m\bigr) \\
&\quad = \left(\left(\prod_{j<k}\sinh\pi(\xi_j-\xi_k)/\zeta\right)^{2s} \cdot \prod_{\alpha=1}^{2s}\prod_{j<k}\sinh\bigl\{\pi(\mu_{2s(a_{(j,\alpha)}-1)+\alpha} - \mu_{2s(a_{(k,\alpha)}-1)+\alpha})/\zeta\bigr\}\right) \\
&\qquad \times \left((2i\zeta)^{2sm}\prod_{j=1}^{2sm}\prod_{b=1}^{m}\sinh\pi(\mu_j-\xi_b-\eta/2)/\zeta\right)^{-1}.
\end{aligned} \tag{5.17}$$

Making use of (5.17) we show that such factors in the denominator of (4.30) that vanish in the limit of sending ξ_p to zero are canceled by the factors in the numerator of (5.17). We thus have shown that the EFP for the finite system, $\tau_{N_s}^{(2s)}(m; \{\xi_p\}_m)$, is continuous with respect to ξ_p at $\xi_p = 0$.

Therefore, expression (5.13) gives the spin-s EFP for the homogeneous chain. That is, we have the following equality:

$$\lim_{\xi_1\to 0}\lim_{\xi_2\to 0}\cdots\lim_{\xi_m\to 0}\bigl(\tau_{\infty}^{(2s)}\bigl(m; \{\xi_p\}_m\bigr)\bigr) = \lim_{N_s\to\infty}\tau_{N_s}^{(2s)}\bigl(m; \{\xi_p = 0\}_{N_s}\bigr). \tag{5.18}$$

5.4. The spin-1 EFP with $m = 1$

Let us calculate $\tau^{(2s)}(m)$ for $s = 1$ and $m = 1$. From formula (5.1) we have

$$
\begin{aligned}
&\tau^{(2)}(1)\\
&= \frac{1}{\sinh\eta}\left(\int_{-\infty-\eta/2}^{\infty-\eta/2} d\lambda_1 + \int_{-\infty-3\eta/2}^{\infty-3\eta/2} d\lambda_1\right)\left(\int_{-\infty-\eta/2}^{\infty-\eta/2} d\lambda_2 + \int_{-\infty-3\eta/2}^{\infty-3\eta/2} d\lambda_2\right)\\
&\quad \times H^{(2)}(\lambda_1,\lambda_2)\det S(\lambda_1,\lambda_2)\\
&= -\frac{1}{i\sin\zeta}\frac{1}{4\zeta^2}\\
&\quad \times \int_{-\infty}^{\infty} d\mu_1 \int_{-\infty}^{\infty} d\mu_2 \frac{\sinh(\mu_1-\xi_1+\eta/2)\sinh(\mu_2-\xi_1-\eta/2)}{\sinh(\mu_1-\mu_2+i\epsilon)\cosh(\pi(\mu_1-\xi_1)/\zeta)\cosh(\pi(\mu_2-\xi_1)/\zeta)}\\
&\quad - \frac{1}{i\sin\zeta}\frac{(-1)}{4\zeta^2}\\
&\quad \times \int_{-\infty}^{\infty} d\mu_1 \int_{-\infty}^{\infty} d\mu_2 \frac{\sinh(\mu_1-\xi_1-\eta/2)\sinh(\mu_2-\xi_1+\eta/2)}{\sinh(\mu_1-\mu_2-2\eta)\cosh(\pi(\mu_1-\xi_1)/\zeta)\cosh(\pi(\mu_2-\xi_1)/\zeta)}.
\end{aligned}
\tag{5.19}
$$

Here we note that $\det S(\lambda_1,\lambda_2) = 0$ for $\lambda_1 = \mu_1 - \eta/2$ and $\lambda_2 = \mu_2 - 3\eta/2$, or for $\lambda_1 = \mu_1 - 3\eta/2$ and $\lambda_2 = \mu_2 - \eta/2$. Showing the following relations of integrals

$$
\begin{aligned}
&\int_{-\infty}^{\infty} d\mu \frac{1}{\cosh(\pi\mu/\zeta)}\frac{\sinh(\mu-\eta/2)}{\sinh(\lambda-\mu+i\epsilon)}\\
&\quad = -\int_{-\infty}^{\infty} d\mu \frac{1}{\cosh(\pi\mu/\zeta)}\frac{\sinh(\mu+\eta/2)}{\sinh(\lambda-\mu-\eta)} - 2\pi i\frac{\sinh(\lambda-\eta/2)}{\cosh(\pi\lambda/\zeta)},\\
&\int_{-\infty}^{\infty} d\lambda \frac{1}{\cosh(\pi\lambda/\zeta)}\frac{\sinh(\lambda+\eta/2)}{\sinh(\lambda-\mu-\eta)} = -\int_{-\infty}^{\infty} d\lambda \frac{1}{\cosh(\pi\lambda/\zeta)}\frac{\sinh(\lambda-\eta/2)}{\sinh(\lambda-\mu-2\eta)},
\end{aligned}
\tag{5.20}
$$

we thus have

$$
\tau^{(2)}(1) = \frac{\pi}{4}\frac{1}{\zeta\sin\zeta}\left(\int_{-\infty}^{\infty} dx \frac{\cosh 2\zeta x - \cosh\eta}{\cosh^2\pi x}\right). \tag{5.21}
$$

Evaluating the integral we obtain the spin-1 EFP with $m = 1$ as follows:

$$
\tau^{(2)}(1) = \frac{\zeta - \sin\zeta\cos\zeta}{2\zeta\sin^2\zeta}. \tag{5.22}
$$

Let us denote by $\langle E_1^{a,b}\rangle$ the ground-state expectation value of operator $E_1^{a,b}$. For the spin-1 case, we have $\widetilde{E}_1^{0,0\,(2+)} + \widetilde{E}_1^{1,1\,(2+)} + \widetilde{E}_1^{2,2\,(2+)} = \widetilde{P}_1^{(2)}$, and hence we have

$$\langle \widetilde{E}_1^{0,0\,(2+)} \rangle + \langle \widetilde{E}_1^{1,1\,(2+)} \rangle + \langle \widetilde{E}_1^{2,2\,(2+)} \rangle = 1. \tag{5.23}$$

Due to the uniaxial symmetry we have $\langle \widetilde{E}_1^{0,0\,(2+)} \rangle = \langle \widetilde{E}_1^{2,2\,(2+)} \rangle$. Thus, we obtain

$$\langle \widetilde{E}_1^{1,1\,(2+)} \rangle = \frac{\cos\zeta(\sin\zeta - \zeta\cos\zeta)}{\zeta\sin^2\zeta}. \tag{5.24}$$

In the XXX limit, we have

$$\lim_{\zeta\to 0} \frac{\zeta - \sin\zeta\cos\zeta}{2\zeta\sin^2\zeta} = \frac{1}{3}. \tag{5.25}$$

The limiting value 1/3 coincides with the spin-1 XXX result obtained by Kitanine [13]. As pointed out in Ref. [13], $\langle E_1^{22} \rangle = \langle E_1^{11} \rangle = \langle E_1^{00} \rangle = 1/3$ for the XXX case since it has the rotational symmetry.

In the symmetric expression of the spin-1 EFP with $m = 1$, putting $\lambda_1 = \mu_1 - \eta/2$ and $\lambda_2 = \mu_2 - 3\eta/2$ in (5.13), we directly obtain the following:

$$\begin{aligned}\tau^{(2s)}(1) = {} & \frac{1}{i\sin\zeta}\frac{1}{4\zeta^2}\int_{-\infty}^{\infty} d\mu_1 \int_{-\infty}^{\infty} d\mu_2 \frac{\sinh(\mu_1 + \eta/2)\sinh(\mu_2 - \eta/2)}{\cosh(\pi\mu_1/\zeta)\cosh(\pi\mu_2/\zeta)} \\ & \times \left(\frac{1}{\sinh(\mu_2 - \mu_1 - i\epsilon)} - \frac{1}{\sinh(\mu_1 - \mu_2 + 2\eta)} \right).\end{aligned} \tag{5.26}$$

In the second line of (5.26) the first term corresponds to the first term of (5.19), while the second term to the second term of (5.19) with μ_1 and μ_2 being exchanged.

6. Spin-s XXZ correlation functions near AF point

6.1. Finite-size correlation functions of the integrable spin-s XXZ spin chain

We now calculate correlation functions other than EFP for the spin-s XXZ spin chain by the method of Section 3.3, making use of the formulas of Hermitian elementary matrices such as (3.18).

We define the correlation function of the spin-2s XXZ spin chain for a given product of $(2s+1)\times(2s+1)$ elementary matrices such as $\widetilde{E}_1^{i_1,j_1(2s+)} \cdots \widetilde{E}_m^{i_m,j_m(2s+)}$ as follows:

$$F^{(2s+)}(\{i_k, j_k\}) = \langle \psi_g^{(2s+)} | \prod_{k=1}^{m} \widetilde{E}_k^{i_k,j_k\,(2s+)} | \psi_g^{(2s+)} \rangle / \langle \psi_g^{(2s+)} | \psi_g^{(2s+)} \rangle. \tag{6.1}$$

By the method of expressing spin-s local operators in terms of spin-1/2 global operators [15], we express the mth product of $(2s+1)\times(2s+1)$ elementary matrices in terms of a $2sm$th product of 2×2 elementary matrices with entries $\{\epsilon_j, \epsilon'_j\}$ as follows:

$$\prod_{k=1}^{m} \widetilde{E}_k^{i_k,j_k\,(2s+)} = C(\{i_k, j_k\}; \{\epsilon'_j, \epsilon_j\}) \widetilde{P}_{12\cdots L}^{(2s)} \cdot \prod_{k=1}^{2sm} e_k^{\epsilon'_k,\epsilon_k} \cdot \widetilde{P}_{12\cdots L}^{(2s)}. \tag{6.2}$$

We evaluate the spin-2s XXZ correlation function $F^{(2s+)}(\{i_k, j_k\})$ by

$$F^{(2s+)}\big(\{i_k, j_k\}\big) = C\big(\{i_k, j_k\}; \big\{\epsilon'_j, \epsilon_j\big\}\big)\big\langle\psi_g^{(2s+)}\big|\widetilde{P}^{(2s)}_{12\cdots L} \times \prod_{j=1}^{2sm} e_j^{\epsilon'_j, \epsilon_j} \cdot \widetilde{P}^{(2s)}_{12\cdots L}\big|\psi_g^{(2s+)}\big\rangle / \big\langle\psi_g^{(2s+)}\big|\psi_g^{(2s+)}\big\rangle. \tag{6.3}$$

We denote the right-hand side of (6.3) by $F^{(2s+)}(\{\epsilon_j, \epsilon'_j\})$.

Let us introduce some symbols (see also [5,54]). We denote by $\boldsymbol{\alpha}^+$ the set of suffices j such that $\epsilon_j = 0$, and by $\boldsymbol{\alpha}^-$ the set of suffices j such that $\epsilon'_j = 1$:

$$\boldsymbol{\alpha}^+ = \{j; \epsilon_j = 0\}, \qquad \boldsymbol{\alpha}^- = \big\{j; \epsilon'_j = 1\big\}. \tag{6.4}$$

We denote by α_+ and α_- the number of elements of the set $\boldsymbol{\alpha}^+$ and $\boldsymbol{\alpha}^-$, respectively. Due to charge conservation, we have

$$\alpha_+ + \alpha_- = 2sm. \tag{6.5}$$

We denote by $j_{\min}$ and $j_{\max}$ the smallest element and the largest element of $\boldsymbol{\alpha}^-$, respectively. We also denote by $j'_{\min}$ and $j'_{\max}$ the smallest element and the largest element of $\boldsymbol{\alpha}^+$, respectively.

Let c_j ($j \in \boldsymbol{\alpha}^-$) and c'_j ($j \in \boldsymbol{\alpha}^+$) be integers such that $1 \leqslant c_j \leqslant M$ for $j \in \boldsymbol{\alpha}^-$ and $1 \leqslant c'_j \leqslant M + j$ for $j \in \boldsymbol{\alpha}^+$. We define sequence $(b_\ell)_{2sm}$ by

$$(b_1, b_2, \ldots, b_{2sm}) = \big(c'_{j'_{\max}}, \ldots, c'_{j'_{\min}}, c_{j_{\min}}, \ldots, c_{j_{\max}}\big). \tag{6.6}$$

Here sequence $(c'_{j'_{\max}}, \ldots, c'_{j'_{\min}}, c_{j_{\min}}, \ldots, c_{j_{\max}})$ is given by the composite of sequence of c'_j's in decreasing order with respect to suffix j, and sequence of c_j's in increasing order with respect to suffix j.

Let us introduce the following symbols:

$$\prod_{j\in\boldsymbol{\alpha}^-}\left(\sum_{c_j=1}^{M}\right)\prod_{j\in\boldsymbol{\alpha}^+}\left(\sum_{c'_j=1}^{M+j}\right) = \sum_{c_{j_{\min}}=1}^{M}\cdots\sum_{c_{j_{\max}}=1}^{M}\sum_{c'_{j'_{\min}}=1}^{M+j'_{\min}}\cdots\sum_{c'_{j'_{\max}}=1}^{M+j'_{\max}}. \tag{6.7}$$

Extending the derivation of the spin-s EFP we can rigorously derive the following expression of the spin-s XXZ correlation functions in the massless regime with $0 \leqslant \zeta < \pi/2s$

$$\begin{aligned} &F^{(2s+)}\big(\big\{\epsilon_j, \epsilon'_j\big\}\big) \\ &= \prod_{j\in\boldsymbol{\alpha}^-}\left(\sum_{c_j=1}^{M}\right)\prod_{j\in\boldsymbol{\alpha}^+}\left(\sum_{c'_j=1}^{M+j}\right)\det M^{(2sm)}\big((b_\ell)_{2sm}\big) \\ &\quad\times(-1)^{\alpha_+}\frac{\prod_{j\in\boldsymbol{\alpha}^-}(\prod_{k=1}^{j-1}\varphi(\lambda_{c_j}-w_k^{(2s)}+\eta)\prod_{k=j+1}^{2sm}\varphi(\lambda_{c_j}-w_k^{(2s)}))}{\prod_{1\leqslant k<\ell\leqslant 2sm}\varphi(\lambda_{b_\ell}-\lambda_{b_k}+\eta)} \\ &\quad\times\frac{\prod_{j\in\boldsymbol{\alpha}^+}(\prod_{k=1}^{j-1}\varphi(\lambda_{c'_j}-w_k^{(2s)}-\eta)\prod_{k=j+1}^{2sm}\varphi(\lambda_{c'_j}-w_k^{(2s)}))}{\prod_{1\leqslant k<\ell\leqslant 2sm}\varphi(w_k^{(2s)}-w_\ell^{(2s)})} + O(1/N_s). \end{aligned} \tag{6.8}$$

Here $\varphi(\lambda) = \sinh\lambda$. We have defined the $2sm \times 2sm$ matrix $M^{(2sm)}((b_j)_{2sm})$ as follows. For $\ell, k = 1, 2, \ldots, 2sm$, the matrix element of (ℓ, k) is given by

$$\left(M^{(2sm)}\left((b_j)_{2sm}\right)\right)_{\ell,k} = \begin{cases} -\delta_{b_\ell - M,k} & \text{for } b_\ell > M, \\ \delta_{\beta(b_\ell),\beta(k)} \cdot \rho(\lambda_{b_\ell} - w_k^{(2s)} + \eta/2)/N_s \rho_{\text{tot}}(\mu_{a(b_\ell)}) & \text{for } b_\ell \leqslant M \end{cases} \tag{6.9}$$

where μ_j denote the centers of λ_j as follows:

$$\lambda_j = \mu_j - \left(\beta(j) - \frac{1}{2}\right)\eta, \quad j = 1, 2, \ldots, 2sm. \tag{6.10}$$

We recall that $a(j)$ and $\beta(j)$ have been defined in terms of the Gauss' symbol $[\cdot]$ by $a(j) = [(j-1)/2s] + 1$ and $\beta(j) = j - 2s[(j-1)/2s]$, respectively.

We remark that under the limit of sending ϵ to zero, the sum over variable c_j is restricted up to M.

6.2. Multiple-integral representations of the spin-s XXZ correlation function for an arbitrary product of elementary matrices

Let us formulate matrix S for the correlation function of an arbitrary product of elementary matrices. We define the (j,k) element of matrix $S = S((\lambda_j)_{2sm}; (w_j^{(2s)})_{2sm})$ by

$$S_{j,k} = \rho\left(\lambda_j - w_k^{(2s)} + \eta/2\right)\delta\left(\alpha(\lambda_j), \beta(k)\right), \quad \text{for } j, k = 1, 2, \ldots, 2sm. \tag{6.11}$$

Here $\delta(\alpha, \beta)$ denotes the Kronecker delta. We define $\alpha(\lambda_j)$ by $\alpha(\lambda_j) = \gamma$ if $\lambda_j = \mu_j - (\gamma - 1/2)\eta$ or $\lambda_j = w_k^{(2s)}$ where $\beta(k) = \gamma$ $(1 \leqslant \gamma \leqslant 2s)$. We remark that μ_j correspond to the centers of complete $2s$-strings λ_j. We also remark that the above definition of matrix S generalizes that of (5.3) since $\alpha(\lambda_j)$ is now also defined also for $\lambda_j = w_k^{(2s)}$.

Let Γ_j be a small contour rotating counterclockwise around $\lambda = w_j^{(2s)}$. Since the $\det S$ has simple poles at $\lambda = w_j^{(2s)}$ with residue $1/2\pi i$, we therefore have

$$\int_{-\infty+i\epsilon}^{\infty+i\epsilon} \det S\left((\lambda_k)_{2sm}\right) d\lambda_1 = \int_{-\infty-i\epsilon}^{\infty-i\epsilon} \det S\left((\lambda_k)_{2sm}\right) d\lambda_1 - \oint_{\Gamma_1} \det S\left((\lambda_k)_{2sm}\right) d\lambda_1. \tag{6.12}$$

For sets $\boldsymbol{\alpha}^-$ and $\boldsymbol{\alpha}^+$ we define $\tilde{\lambda}_j$ for $j \in \boldsymbol{\alpha}^-$ and $\tilde{\lambda}'_j$ for $j \in \boldsymbol{\alpha}^+$, respectively, by the following sequence:

$$\left(\tilde{\lambda}'_{j'_{\max}}, \ldots, \tilde{\lambda}'_{j'_{\min}}, \tilde{\lambda}_{j_{\min}}, \tilde{\lambda}_{j_{\max}}\right) = (\lambda_1, \ldots, \lambda_{2sm}). \tag{6.13}$$

Thus, from the expression of the correlation function in terms of a finite sum (6.8) we obtain the multiple-integral representation as follows:

$$\begin{aligned} & F^{(2s+)}\left(\{\epsilon_j, \epsilon'_j\}\right) \\ & = \left(\int_{-\infty+i\epsilon}^{\infty+i\epsilon} + \cdots + \int_{-\infty-i(2s-1)\zeta+i\epsilon}^{\infty-i(2s-1)\zeta+i\epsilon}\right) d\lambda_1 \cdots \left(\int_{-\infty+i\epsilon}^{\infty+i\epsilon} + \cdots + \int_{-\infty-i(2s-1)\zeta+i\epsilon}^{\infty-i(2s-1)\zeta+i\epsilon}\right) d\lambda_{\alpha_+} \\ & \quad \times \left(\int_{-\infty-i\epsilon}^{\infty-i\epsilon} + \cdots + \int_{-\infty-i(2s-1)\zeta-i\epsilon}^{\infty-i(2s-1)\zeta-i\epsilon}\right) d\lambda_{\alpha_++1} \cdots \end{aligned}$$

$$\times\left(\int_{-\infty-i\epsilon}^{\infty-i\epsilon}+\cdots+\int_{-\infty-i(2s-1)\zeta-i\epsilon}^{\infty-i(2s-1)\zeta-i\epsilon}\right)d\lambda_m$$
$$\times Q\big(\{\epsilon_j,\epsilon'_j\};\lambda_1,\ldots,\lambda_{2sm}\big)\det S(\lambda_1,\ldots,\lambda_{2sm}). \tag{6.14}$$

Here we have defined $Q(\{\epsilon_j,\epsilon'_j\};\lambda_1,\ldots,\lambda_{2sm})$ in terms of small numbers $\epsilon_{\ell,k}$ of (5.4) by

$$Q\big(\{\epsilon_j,\epsilon'_j\};\lambda_1,\ldots,\lambda_{2sm}\big)$$
$$=(-1)^{\alpha_+}\frac{\prod_{j\in\boldsymbol{\alpha}^-}(\prod_{k=1}^{j-1}\varphi(\tilde\lambda_j-w_k^{(2s)}+\eta)\prod_{k=j+1}^{2sm}\varphi(\tilde\lambda_j-w_k^{(2s)}))}{\prod_{1\leqslant k<\ell\leqslant 2sm}\varphi(\lambda_\ell-\lambda_k+\eta+\epsilon_{\ell,k})}$$
$$\times\frac{\prod_{j\in\boldsymbol{\alpha}^+}(\prod_{k=1}^{j-1}\varphi(\tilde\lambda'_j-w_k^{(2s)}-\eta)\prod_{k=j+1}^{2sm}\varphi(\tilde\lambda'_j-w_k^{(2s)}))}{\prod_{1\leqslant k<\ell\leqslant 2sm}\varphi(w_k^{(2s)}-w_\ell^{(2s)})}. \tag{6.15}$$

Thus, correlation functions (6.1) are expressed in the form of a single term of multiple integrals.

Similarly as the symmetric spin-s EFP, we can show the symmetric expression of the multiple-integral representations of the spin-s correlation function as follows:

$$F^{(2s+)}\big(\{\epsilon_j,\epsilon'_j\}\big)$$
$$=\frac{1}{\prod_{1\leqslant\alpha<\beta\leqslant 2s}\sinh^m(\beta-\alpha)\eta}\prod_{1\leqslant k<l\leqslant m}\frac{\sinh^{2s}(\pi(\xi_k-\xi_l)/\zeta)}{\prod_{j=1}^{2s}\prod_{r=1}^{2s}\sinh(\xi_k-\xi_l+(r-j)\eta)}$$
$$\times\sum_{\sigma\in\mathcal{S}_{2sm}/(\mathcal{S}_m)^{2s}}\prod_{j=1}^{\alpha_+}\left(\int_{-\infty+i\epsilon}^{\infty+i\epsilon}+\cdots+\int_{-\infty-i(2s-1)\zeta+i\epsilon}^{\infty-i(2s-1)\zeta+i\epsilon}\right)d\mu_{\sigma j}$$
$$\times\prod_{j=\alpha_++1}^{2sm}\left(\int_{-\infty-i\epsilon}^{\infty-i\epsilon}+\cdots+\int_{-\infty-i(2s-1)\zeta-i\epsilon}^{\infty-i(2s-1)\zeta-i\epsilon}\right)d\mu_{\sigma j}$$
$$\times(\operatorname{sgn}\sigma)Q\big(\{\epsilon_j,\epsilon'_j\};\lambda_{\sigma 1},\ldots,\lambda_{\sigma(2sm)}\big)\left(\prod_{j=1}^{2sm}\frac{\prod_{b=1}^m\prod_{\beta=1}^{2s-1}\sinh(\lambda_j-\xi_b+\beta\eta)}{\prod_{b=1}^m\cosh(\pi(\mu_j-\xi_b)/\zeta)}\right)$$
$$\times\frac{i^{2sm^2}}{(2i\zeta)^{2sm}}\prod_{\gamma=1}^{2s}\prod_{1\leqslant b<a\leqslant m}\sinh\big(\pi(\mu_{2s(a-1)+\gamma}-\mu_{2s(b-1)+\gamma})/\zeta\big).$$

It is straightforward to take the homogeneous limit: $\xi_k\to 0$. Here we recall that $(\operatorname{sgn}\sigma)$ denotes the sign of permutation $\sigma\in\mathcal{S}_{2sm}/(\mathcal{S}_m)^{2s}$.

6.3. An example of the spin-1 correlation function

Applying formula (3.18) to the spin-1 case with $m=n=1$, we have

$$\widetilde{E}_1^{1,1\,(2+)}=2\widetilde{P}_{1\cdots L}^{(2s)}D^{(1+)}(w_1)A^{(1+)}(w_2)\prod_{\alpha=3}^{2N_s}\big(A^{(1+)}+D^{(1+)}\big)(w_\alpha)\widetilde{P}_{1\cdots L}^{(2s)}. \tag{6.16}$$

Therefore, we evaluate it sending ϵ to zero, as follows:

$$
\begin{aligned}
&\langle \psi_g^{(2+)} | \widetilde{E}_1^{1,1\,(2+)} | \psi_g^{(2+)} \rangle / \langle \psi_g^{(2+)} | \psi_g^{(2+)} \rangle \\
&\quad = 2 \lim_{\epsilon \to 0} \langle \psi_g^{(2+;\epsilon)} | D^{(2+;\epsilon)} \big(w_1^{(2;\epsilon)} \big) A^{(2+;\epsilon)} \big(w_2^{(2;\epsilon)} \big) \\
&\qquad \times \prod_{\alpha=3}^{2N_s} \big(A^{(2+;\epsilon)} + D^{(2+;\epsilon)} \big) \big(w_\alpha^{(2;\epsilon)} \big) | \psi_g^{(2+;\epsilon)} \rangle / \langle \psi_g^{(2+;\epsilon)} | \psi_g^{(2+;\epsilon)} \rangle \\
&\quad = 2 \left(\int_{-\infty+i\epsilon}^{\infty+i\epsilon} + \int_{-\infty-i\zeta+i\epsilon}^{\infty-i\zeta+i\epsilon} \right) d\lambda_1 \left(\int_{-\infty-i\epsilon}^{\infty-i\epsilon} + \int_{-\infty-i\zeta-i\epsilon}^{\infty-i\zeta-i\epsilon} \right) d\lambda_2 \, Q(\lambda_1, \lambda_2) \, \det S(\lambda_1, \lambda_2)
\end{aligned} \tag{6.17}
$$

where $Q(\lambda_1, \lambda_2)$ is given by

$$
Q(\lambda_1, \lambda_2) = (-1) \frac{\varphi(\lambda_2 - \xi_1 + \eta) \varphi(\lambda_1 - \xi_1 - \eta)}{\varphi(\lambda_2 - \lambda_1 + \eta + \epsilon_{2,1}) \varphi(\eta)} \tag{6.18}
$$

and matrix $S(\lambda_1, \lambda_2)$ is given by

$$
S(\lambda_1, \lambda_2) = \begin{pmatrix} \rho(\lambda_1 - w_1^{(2)} + \eta/2) \delta(\lambda_1, 1) & \rho(\lambda_1 - w_2^{(2)} + \eta/2) \delta(\lambda_1, 2) \\ \rho(\lambda_2 - w_1^{(2)} + \eta/2) \delta(\lambda_2, 1) & \rho(\lambda_2 - w_2^{(2)} + \eta/2) \delta(\lambda_2, 2) \end{pmatrix}. \tag{6.19}
$$

We thus note that the correlation function is now expressed in terms of a single product of the multiple-integral representation.

Let us now evaluate the double integral (6.17), explicitly. The integral over λ_1 is decomposed into the following:

$$
\begin{aligned}
&\left(\int_{-\infty+i\epsilon}^{\infty+i\epsilon} + \int_{-\infty-i\zeta+i\epsilon}^{\infty-i\zeta+i\epsilon} \right) d\lambda_1 \, Q(\lambda_1, \lambda_2) \det S(\lambda_1, \lambda_2) \\
&\quad = \left(\int_{-\infty-i\zeta/2}^{\infty-i\zeta/2} + \int_{-\infty-i3\zeta/2}^{\infty-i3\zeta/2} \right) d\lambda_1 \, Q(\lambda_1, \lambda_2) \det S(\lambda_1, \lambda_2) \\
&\qquad - \oint_{\Gamma_1} d\lambda_1 \, Q(\lambda_1, \lambda_2) \det S(\lambda_1, \lambda_2) - \oint_{\Gamma_2} d\lambda_1 \, Q(\lambda_1, \lambda_2) \det S(\lambda_1, \lambda_2).
\end{aligned} \tag{6.20}
$$

Thus, the integral (6.17) is calculated as

$$
\begin{aligned}
&\langle \widetilde{\psi}_g^{(2+)} | \widetilde{E}_1^{11\,(2+)} | \widetilde{\psi}_g^{(2+)} \rangle / \big(2 \langle \widetilde{\psi}_g^{(2+)} | \widetilde{\psi}_g^{(2+)} \rangle \big) \\
&\quad = -2\pi i \int_{-\infty}^{\infty} \frac{\sinh(x - \eta/2) \sinh(x - 3\eta/2)}{\sinh \eta} \rho^2(x) \, dx \\
&\qquad + 2 \cosh \eta \int_{-\infty}^{\infty} \frac{\sinh(x - \eta/2)}{\sinh(x + \eta/2)} \rho(x) \, dx - \int_{-\infty}^{\infty} \rho(x) \, dx \\
&\qquad + (-1) 2 \cosh \eta \int_{-\infty}^{\infty} \frac{\sinh(x - \eta/2)}{\sinh(x + \eta/2)} \rho(x) \, dx = \frac{\cos \zeta (\sin \zeta - \zeta \cos \zeta)}{2\zeta \sin^2 \zeta}.
\end{aligned} \tag{6.21}
$$

We have thus confirmed (5.24) directly evaluating the integrals.

7. Concluding remarks

In the paper we have explicitly shown the multiple-integral representation of the emptiness formation probability for the integrable spin-s XXZ spin chain in a region of the massless regime of $\eta = i\zeta$ with $0 \leqslant \zeta < \pi/2s$. We have also calculated the emptiness formation probability for the homogeneous case of the integrable spin-s XXZ spin chain.

In the XXX limit where we send ζ to zero, the expression of EFP for the spin-s XXZ case reduces to that of the spin-s XXX case.

Moreover, we have presented a formula for the multiple-integral representation of the spin-s XXZ correlation function of an arbitrary product of elementary matrices in the massless regime where $\eta = i\zeta$ with $0 \leqslant \zeta < \pi/2s$. We have also presented the symmetric expression of the multiple-integral representations of the spin-s XXZ correlation functions.

Finally, we have introduced conjugate vectors $\widetilde{\|2s,n\rangle}$ in order to formulate Hermitian elementary matrices $\widetilde{E}^{m,n\,(2s+)}$ and Hermitian projection operators $\widetilde{P}^{(\ell)}$ in the massless regime. We have also defined the massless fusion R-matrices $\widetilde{R}^{(\ell,2s\,w)}$ for $w=+$ and p.

Acknowledgements

One of the authors (T.D.) would like to thank B.M. McCoy and other participants of the workshop on the integrable chiral Potts model, organized by M.T. Batchelor, Kioloa, NSW, Australia, December 7–11, 2008, for many useful comments. He is grateful to K. Motegi, J. Sato, and Y. Takeyama for bringing him useful references. Furthermore, the authors would like to thank S. Miyashita for encouragement and keen interest in this work. This work is partially supported by Grant-in-Aid for Scientific Research (C) No. 20540365.

Appendix A. Affine quantum group with homogeneous grading

The affine quantum algebra $U_q(\widehat{sl_2})$ is an associative algebra over $\mathbf{C}$ generated by $X_i^{\pm}$, $K_i^{\pm}$ for $i=0,1$ with the following relations:

$$\begin{aligned}
&K_i K_i^{-1} = K_i^{-1} K_i = 1, \qquad K_i X_i^{\pm} K_i^{-1} = q^{\pm 2} X_i^{\pm}, \qquad K_i X_j^{\pm} K_i^{-1} = q^{\mp 2} X_j^{\pm} \quad (i \neq j),\\
&\left[X_i^{+}, X_j^{-}\right] = \delta_{i,j}\frac{K_i - K_i^{-1}}{q - q^{-1}},\\
&\left(X_i^{\pm}\right)^3 X_j^{\pm} - [3]_q \left(X_i^{\pm}\right)^2 X_j^{\pm} X_i^{\pm} + [3]_q X_i^{\pm} X_j^{\pm}\left(X_i^{\pm}\right)^2 - X_j^{\pm}\left(X_i^{\pm}\right)^3 = 0 \quad (i \neq j).
\end{aligned} \tag{A.1}$$

Here the symbol $[n]_q$ denotes the q-integer of an integer n:

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}. \tag{A.2}$$

The algebra $U_q(\widehat{sl_2})$ is also a Hopf algebra over $\mathbf{C}$ with comultiplication

$$\begin{aligned}
&\Delta\left(X_i^{+}\right) = X_i^{+} \otimes 1 + K_i \otimes X_i^{+}, \qquad \Delta\left(X_i^{-}\right) = X_i^{-} \otimes K_i^{-1} + 1 \otimes X_i^{-},\\
&\Delta(K_i) = K_i \otimes K_i,
\end{aligned} \tag{A.3}$$

and antipode: $S(K_i) = K_i^{-1}$, $S(X_i) = -K_i^{-1} X_i^{+}$, $S(X_i^{-}) = -X_i^{-} K_i$, and count: $\varepsilon(X_i^{\pm}) = 0$ and $\varepsilon(K_i) = 1$ for $i = 0, 1$.

The algebra $U_q(sl_2)$ is given by the Hopf subalgebra of $U_q(\widehat{sl_2})$ generated by $X_i^{\pm}$, K_i with either $i=0$ or $i=1$. Hereafter we denote by $X^{\pm}$ and K the generators of $U_q(sl_2)$.

For a given complex number λ we define a homomorphism of algebras $\varphi_\lambda : U_q(\widehat{sl_2}) \to U_q(sl_2)$

$$\varphi_\lambda\left(X_0^{\pm}\right) = e^{\pm 2\lambda} X^{\mp}, \qquad \varphi_\lambda\left(X_1^{\pm}\right) = X^{\pm}, \qquad \varphi_\lambda(K_0) = K^{-1}, \qquad \varphi_\lambda(K_1) = K. \tag{A.4}$$

Map (A.4) is associated with homogeneous grading [8]. For a representation $(\pi, V^{(\ell)})$ of $U_q(sl_2)$ we have a representation of $U_q(\widehat{sl_2})$ by $\pi(\varphi_\lambda(a))$ for $a \in U_q(\widehat{sl_2})$. We call it the spin-$\ell/2$ evaluation representation with evaluation parameter λ, and denote it by $(\pi_\lambda, V^{(\ell)}(\lambda))$ or $V^{(\ell)}(\lambda)$.

We define opposite coproduct Δ^{op} by

$$\Delta^{op}(a) = \tau \circ \Delta(a) \quad \text{for } a \in U_q(sl_2), \tag{A.5}$$

where τ denotes the permutation operator: $\tau(a \otimes b) = b \otimes a$ for $a, b \in U_q(sl_2)$.

Appendix B. Fusion projection operators being idempotent

We give the derivation [23] of $(P_{12\cdots\ell}^{(\ell)})^2 = P_{12\cdots\ell}^{(\ell)}$ making use of the Yang–Baxter equations.

Lemma B.1. *Operators $P_{12\cdots\ell}^{(\ell)}$ defined by (2.12) have the following two expressions*:

$$P_{12\cdots\ell-1}^{(\ell-1)} \check{R}_{\ell-1,\ell}^{+}\big((\ell-1)\eta\big) P_{12\cdots\ell-1}^{(\ell-1)} = P_{23\cdots\ell}^{(\ell-1)} \check{R}_{1,2}^{+}\big((\ell-1)\eta\big) P_{23\cdots\ell}^{(\ell-1)}. \tag{B.1}$$

Proof. Applying notation (2.2) to permutation operator $\Pi_{1,2}$ we define permutation operators $\Pi_{j,k}$ for integers j and k satisfying $0 \leqslant j < k \leqslant L$. The form of the left-hand side of (B.1) is expressed in terms of R-matrices as follows (see also Eq. (3.7) of [15])

$$P_{1\cdots\ell}^{(\ell)} = \prod_{j=1}^{[\ell/2]} \Pi_{j,\ell-j+1} \cdot R_{\ell-1\,\ell}^{+} \cdots R_{2,3\cdots\ell}^{+} R_{1,2\cdots\ell}^{+}. \tag{B.2}$$

Making use of the Yang–Baxter equations (2.4) we reformulate (B.2) as follows:

$$P_{1\cdots\ell}^{(\ell)} = \prod_{j=1}^{[\ell/2]} \Pi_{j,\ell-j+1} \cdot R_{12}^{+} R_{12,3}^{+} \cdots R_{1,2\cdots\ell}^{+} \tag{B.3}$$

which gives the expression of the right-hand side of (B.1). $\square$

From (B.1) we show that $P_{j+1\,j+2\cdots j+\ell}^{(\ell)}$ is expressed as follows:

$$\begin{aligned} &P_{j+1\,j+2\cdots j+\ell-1}^{(\ell-1)} \check{R}_{j+\ell-1,j+\ell}^{+}\big((\ell-1)\eta\big) P_{j+1 j+2\cdots j+\ell-1}^{(\ell-1)} \\ &\quad = P_{j+2\,j+3\cdots j+\ell}^{(\ell-1)} \check{R}_{j+1,j+2}^{+}\big((\ell-1)\eta\big) P_{j+2\,j+3\cdots j+\ell}^{(\ell-1)}. \end{aligned} \tag{B.4}$$

Lemma B.2. *Operator $P_{12\cdots\ell}^{(\ell)}$ projects operator $\check{R}_{\ell-1,\ell}(u)$ to 1 as follows*:

$$P_{12\cdots\ell}^{(\ell)} \check{R}_{\ell-1,\ell}^{+}(u) = P_{12\cdots\ell}^{(\ell)}. \tag{B.5}$$

Proof. Due to the spectral decomposition of the R-matrix, we have $P_{\ell-1,\ell}^{(2)} \check{R}_{\ell-1,\ell}^{+}(u) = P_{\ell-1,\ell}^{(2)}$. Applying (B.1) and (B.4), we thus obtain (B.5). $\square$

Proposition B.3. *Operators* $P^{(\ell)}_{12\cdots\ell}$ *defined by* (2.12) *are idempotent*: $(P^{(\ell)}_{12\cdots\ell})^2 = P^{(\ell)}_{12\cdots\ell}$.

Proof. We show it from (B.5) by induction on ℓ. Suppose that it is idempotent for ℓ. We have

$$\begin{aligned}\left(P^{(\ell+1)}_{12\cdots\ell+1}\right)^2 &= P^{(\ell)}_{12\cdots\ell}R^{+}_{\ell\,\ell+1}(\ell\eta)\cdot\left(P^{(\ell)}_{12\cdots\ell}\right)^2\cdot R^{+}_{\ell\,\ell+1}(\ell\eta)P^{(\ell)}_{12\cdots\ell}\\ &= P^{(\ell)}_{12\cdots\ell}R^{+}_{\ell\,\ell+1}(\ell\eta)\cdot P^{(\ell)}_{12\cdots\ell}R^{+}_{\ell\,\ell+1}(\ell\eta)\cdot P^{(\ell)}_{12\cdots\ell}\\ &= P^{(\ell)}_{12\cdots\ell}R^{+}_{\ell\,\ell+1}(\ell\eta)P^{(\ell)}_{12\cdots\ell}\cdot P^{(\ell)}_{12\cdots\ell} = P^{(\ell)}_{12\cdots\ell}R^{+}_{\ell\,\ell+1}(\ell\eta)P^{(\ell)}_{12\cdots\ell}. \qquad \Box \end{aligned} \tag{B.6}$$

Appendix C. Basis vectors of spin-$\ell/2$ representation of $U_q(sl_2)$

In terms of the q-integer $[n]_q$ defined in (A.2), we define the q-factorial $[n]_q!$ for integers n by

$$[n]_q! = [n]_q[n-1]_q\cdots[1]_q. \tag{C.1}$$

For integers m and n satisfying $m \geqslant n \geqslant 0$ we define the q-binomial coefficients as follows

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{[m]_q!}{[m-n]_q![n]_q!}. \tag{C.2}$$

We now define the basis vectors of the $(\ell+1)$-dimensional irreducible representation of $U_q(sl_2)$, $\|\ell, n\rangle$ for $n = 0, 1, \ldots, \ell$ as follows. We define $\|\ell, 0\rangle$ by

$$\|\ell, 0\rangle = |0\rangle_1 \otimes |0\rangle_2 \otimes \cdots \otimes |0\rangle_\ell. \tag{C.3}$$

Here $|\alpha\rangle_j$ for $\alpha = 0, 1$ denote the basis vectors of the spin-$1/2$ representation defined on the jth position in the tensor product. We define $\|\ell, n\rangle$ for $n \geqslant 1$ and evaluate them as follows [15]

$$\begin{aligned}\|\ell, n\rangle &= \left(\Delta^{(\ell-1)}\left(X^-\right)\right)^n \|\ell, 0\rangle \frac{1}{[n]_q!}\\ &= \sum_{1\leqslant i_1<\cdots<i_n\leqslant\ell} \sigma^-_{i_1}\cdots\sigma^-_{i_n}|0\rangle\, q^{i_1+i_2+\cdots+i_n-n\ell+n(n-1)/2}. \end{aligned} \tag{C.4}$$

We define the conjugate vectors explicitly by the following:

$$\langle \ell, n\| = \begin{bmatrix} \ell \\ n \end{bmatrix}_q^{-1} q^{n(\ell-n)} \sum_{1\leqslant i_1<\cdots<i_n\leqslant\ell} \langle 0|\sigma^+_{i_1}\cdots\sigma^+_{i_n} q^{i_1+\cdots+i_n-n\ell+n(n-1)/2}. \tag{C.5}$$

It is easy to show the normalization conditions [15]: $\langle \ell, n\|\|\ell, n\rangle = 1$. In the massive regime where $q = \exp\eta$ with real η, conjugate vectors $\langle \ell, n\|$ are Hermitian conjugate to vectors $\|\ell, n\rangle$.

Through the recursive construction (2.12) of $P^{(\ell)}$'s, it is easy to show the following [15]:

$$P^{(\ell)}_{12\cdots\ell}\|\ell, n\rangle = \|\ell, n\rangle, \qquad \langle \ell, n\| P^{(\ell)}_{12\cdots\ell} = \langle \ell, n\|. \tag{C.6}$$

Thus, the fusion projector $P^{(\ell)}$ is consistent with the spin-$\ell/2$ representation of $U_q(sl_2)$.

In order to define Hermitian elementary matrices, we now introduce another set of dual basis vectors. For a given nonzero integer ℓ we define $\widetilde{\langle \ell, n\|}$ for $n = 0, 1, \ldots, n$, by

$$\widetilde{\langle \ell, n\|} = \begin{pmatrix} \ell \\ n \end{pmatrix}^{-1} \sum_{1\leqslant i_1<\cdots<i_n\leqslant\ell} \langle 0|\sigma^+_{i_1}\cdots\sigma^+_{i_n} q^{-(i_1+\cdots+i_n)+n\ell-n(n-1)/2}. \tag{C.7}$$

They are conjugate to $\|\ell, n\rangle$: $\widetilde{\langle \ell, m\|}\|\ell, n\rangle = \delta_{m,n}$.

In the massless regime where $|q|=1$, matrix $\|\ell,n\rangle\widetilde{\langle \ell,n\|}$ is Hermitian: $(\|\ell,n\rangle\widetilde{\langle \ell,n\|})^{\dagger} = \|\ell,n\rangle\widetilde{\langle \ell,n\|}$. However, in order to define projection operators $\widetilde{P}$ such that $P\widetilde{P}=P$, we define another set of vectors $\widetilde{\|\ell,n\rangle}$ in Section 2.4. They are conjugate to the dual vectors $\langle \ell,n\|$.

Appendix D. The massless fusion R-matrices of the spin-1 case

Let us evaluate the matrix elements of the massless monodromy matrix $\widetilde{T}_{0,1}^{(1,2+)}(\lambda_0;\xi_1)$, i.e. the massless L operator of the spin-1 representation

$$\widetilde{T}_{0,1}^{(1,2+)}(\lambda_0;\xi_1) = \widetilde{P}_{12}^{(2)} R_{0,12}^{(1,1+)}\big(\lambda_0;\{w_j^{(2)}\}_2\big)\widetilde{P}_{12}^{(2)}. \tag{D.1}$$

Here we have set inhomogeneous parameters $w_1^{(2)}=\xi_1$ and $w_2^{(2)}=\xi_1-\eta$. Let us recall $R_{0,12}^{+} = R_{0,2}^{+}R_{0,1}^{+}$. For instance, we have $A_{12}^{+} = A_2^{+}A_1^{+} + B_2^{+}C_1^{+}$. In terms of $b_{0j} = b(\lambda_0 - w_j^{(2)})$ and $c_{0j} = c(\lambda_0 - w_j^{(2)})$ for $j=1,2$, the $(1,1)$ element of $\widetilde{A}_1^{(2+)}$ is given by

$$\langle 2,1\|A^{(2+)}(\lambda_0)\widetilde{\|2,1\rangle} = \big(b_{01}+b_{02}+q^{-2}c_{01}c_{02}\big)/2. \tag{D.2}$$

Thus, setting $u=\lambda_0-\xi_1$, all the nonzero matrix elements of $\widetilde{T}^{(1,2+)}(\lambda_0)$ are given by

$$\begin{aligned}
&\langle 2,0\|A^{(2+)}(\lambda_0)\widetilde{\|2,0\rangle} = \langle 2,2\|D^{(2+)}(\lambda_0)\widetilde{\|2,2\rangle} = 1,\\
&\langle 2,1\|A^{(2+)}(\lambda_0)\widetilde{\|2,1\rangle} = \langle 2,1\|D^{(2+)}(\lambda_0)\widetilde{\|2,1\rangle} = \sinh(u+\eta)/\sinh(u+2\eta),\\
&\langle 2,2\|A^{(2+)}(\lambda_0)\widetilde{\|2,2\rangle} = \langle 2,0\|D^{(2+)}(\lambda_0)\widetilde{\|2,0\rangle} = \sinh u/\sinh(u+2\eta),\\
&\langle 2,1\|B^{(2+)}(\lambda_0)\widetilde{\|2,0\rangle} = e^{-u}\sinh\eta/\sinh(u+2\eta),\\
&\langle 2,2\|B^{(2+)}(\lambda_0)\widetilde{\|2,1\rangle} = [2]_q\, q^{-1}e^{-u}\sinh\eta/\sinh(u+2\eta),\\
&\langle 2,0\|C^{(2+)}(\lambda_0)\widetilde{\|2,1\rangle} = [2]_q\, e^{u}\sinh\eta/\sinh(u+2\eta),\\
&\langle 2,1\|C^{(2+)}(\lambda_0)\widetilde{\|2,2\rangle} = qe^{u}\sinh\eta/\sinh(u+2\eta).
\end{aligned} \tag{D.3}$$

We should remark that the massive monodromy matrix $T_{0,1}^{(1,2+)}(\lambda_0;\xi_1)$ has the same matrix elements as the massless monodromy matrix $\widetilde{T}_{0,1}^{(1,2+)}(\lambda_0;\xi_1)$. For instance, we calculate the $(1,1)$ element of operator $A_1^{(2+)}$ as follows:

$$\langle 2,1\|A^{(2+)}(\lambda_0)\|2,1\rangle = b_{01}q^{-2}+b_{02}+q^{-2}c_{01}c_{02} = \sinh(u+\eta)/\sinh(u+2\eta). \tag{D.4}$$

Let us define the matrix elements of the massless fusion R matrix of type $(2,2)$ as follows:

$$\widetilde{R}_{0,1}^{(2,2+)}(\lambda_0-\xi_1)_{c_0\,c_1}^{b_0\,b_1} = {}_1\langle 2,b_1\|\,{}_a\langle 2,b_0\|R_{0,1}^{(2,2+)}(\lambda_0-\xi_1)\widetilde{\|2,c_0\rangle}_a\widetilde{\|2,c_1\rangle}_1. \tag{D.5}$$

Here $\|2,c_0\rangle_a$ and $\|2,c_1\rangle_1$ denote vectors in the auxiliary space $V_0^{(2)}$ and the quantum space $V_1^{(2)}$, respectively. Making use of matrix elements of the monodromy matrix of type $(1,2)$ we derive the fusion R matrix of type $(2,2)$. For an illustration, let us calculate $\widetilde{R}_{0,1}^{(2,2+)}(u)_{0\,1}^{1\,0}$

$$\begin{aligned}
&{}_a\langle 2,1\|R_{0,1}^{(2\,2+)}(\lambda_0-\xi_1)\widetilde{\|2,0\rangle}_a\\
&\quad = \big(A_1^{(2+)}(\lambda)C_1^{(2+)}(\lambda-\eta)+q^{-1}C_1^{(2+)}(\lambda)A_1^{(2+)}(\lambda-\eta)\big)q/[2]_q.
\end{aligned} \tag{D.6}$$

Evaluating operators $A_1^{(2+)}$ and $C_1^{(2+)}$ in the quantum space $V_1^{(2)}$, we have

$$R_{0,1}^{(22+)}(u)_{0\,1}^{1\,0} = \frac{[2]_q e^u \sinh\eta}{\sinh(u+2\eta)}. \tag{D.7}$$

The fusion R-matrix becomes permutation $\Pi_{0,1}$ at $u=0$. In fact, we have $R_{0,1}^{(2,2+)}(0)_{0\,1}^{1\,0} = 1$.

Appendix E. Spin-s elementary matrices in global operators

For integers i_k and j_k with $1 \leqslant i_1 < \cdots < i_m \leqslant \ell$ and $1 \leqslant j_1 < \cdots < j_n \leqslant \ell$, we have

$$\begin{aligned}
&\widetilde{\|\ell, m\rangle}\langle \ell, n\| \\
&= \binom{\ell}{n} \begin{bmatrix} \ell \\ m \end{bmatrix}_q \begin{bmatrix} \ell \\ n \end{bmatrix}_q^{-1} q^{-(i_1+\cdots+i_m)+(j_1+\cdots+j_n)+m(m+1)/2-n(n+1)/2} \\
&\quad \times \widetilde{P}_{1\cdots\ell}^{(\ell)} \left(\prod_{k=1}^{m} e_{i_k}^{1,0} \cdot \prod_{p=1;\, p\neq i_1,\ldots,i_m, j_1,\ldots,j_n}^{\ell} e_p^{0,0} \cdot \prod_{q=1}^{n} e_{j_q}^{0,1} \right) \widetilde{P}_{1\cdots\ell}^{(\ell)}.
\end{aligned} \tag{E.1}$$

Applying the spin-1/2 formulas of QISP [4] to (E.1), we can express any given spin-s local operator in terms of the spin-1/2 global operators. It is parallel to the massive case [15]. Let us set $i_1 = 1, i_2 = 2, \ldots, i_m = m$ and $j_1 = 1$, $j_2 = 2, \ldots, j_n = n$ in (E.1). For $m > n$ we have

$$\begin{aligned}
&\widetilde{E}_i^{m,n\,(\ell+)} \\
&= \binom{\ell}{n} \begin{bmatrix} \ell \\ m \end{bmatrix}_q \begin{bmatrix} \ell \\ n \end{bmatrix}_q^{-1} \widetilde{P}_{1\cdots L}^{(\ell)} \prod_{\alpha=1}^{(i-1)\ell} \big(A^{(1+)} + D^{(1+)}\big)(w_\alpha) \prod_{k=1}^{n} D^{(1+)}(w_{(i-1)\ell+k}) \\
&\quad \times \prod_{k=n+1}^{m} B^{(1+)}(w_{(i-1)2s+k}) \prod_{k=m+1}^{\ell} A^{(1+)}(w_{(i-1)\ell+k}) \\
&\quad \times \prod_{\alpha=i\ell+1}^{\ell N_s} \big(A^{(1+)} + D^{(1+)}\big)(w_\alpha) P_{1\cdots L}^{(\ell)}.
\end{aligned} \tag{E.2}$$

For $m < n$ we have

$$\begin{aligned}
&\widetilde{E}_i^{m,n\,(\ell+)} \\
&= \binom{\ell}{n} \begin{bmatrix} \ell \\ m \end{bmatrix}_q \begin{bmatrix} \ell \\ n \end{bmatrix}_q^{-1} \widetilde{P}_{1\cdots L}^{(\ell)} \prod_{\alpha=1}^{(i-1)\ell} \big(A^{(1+)} + D^{(1+)}\big)(w_\alpha) \prod_{k=1}^{m} D^{(1+)}(w_{(i-1)\ell+k}) \\
&\quad \times \prod_{k=m+1}^{n} C^{(1+)}(w_{(i-1)2s+k}) \prod_{k=m+1}^{\ell} A^{(1+)}(w_{(i-1)\ell+k}) \\
&\quad \times \prod_{\alpha=i\ell+1}^{\ell N_s} \big(A^{(1+)} + D^{(1+)}\big)(w_\alpha) P_{1\cdots L}^{(\ell)}.
\end{aligned} \tag{E.3}$$

Appendix F. Derivation of the density of string centers

In terms of shifted rapidities $\tilde{\lambda}_A$ with $A = 2s(a-1)+\alpha$ for $a = 1, 2, \ldots, N_s/2$ and $\alpha = 1, 2, \ldots, 2s$, the Bethe-ansatz equations for the homogeneous chain are given by

$$\left(\frac{\sinh(\tilde{\lambda}_A + s\eta)}{\sinh(\tilde{\lambda}_A - s\eta)}\right)^{N_s} = \prod_{B=1;\, B\neq A}^{M} \frac{\sinh(\tilde{\lambda}_A - \tilde{\lambda}_B + \eta)}{\sinh(\tilde{\lambda}_A - \tilde{\lambda}_B - \eta)}, \quad \text{for } A = 1, 2, \ldots, M. \tag{F.1}$$

Putting $\lambda_A = \mu_a - (2s + 1 - 2\alpha)$ and taking the product over α for $\alpha = 1, 2, \ldots, 2s$, for the left-hand side of (F.1) and for the right-hand side of (F.1), we have

$$(-1)^{2s}\left\{\prod_{k=1}^{2s}\left(\frac{\sinh((k-1/2)\eta - \mu_a)}{\sinh((k-1/2)\eta + \mu_a)}\right)^{N_s}\right\}^{-1} = (-1)^{2s+N_s/2}\prod_{b=1}^{N_s/2}\left\{\frac{\sinh(2s\eta - (\mu_a - \mu_b))}{\sinh(2s\eta + (\mu_a - \mu_b))}\prod_{k=1}^{2s-1}\left(\frac{\sinh(k\eta - (\mu_a - \mu_b))}{\sinh(k\eta + (\mu_a - \mu_b))}\right)^2\right\}^{-1}. \tag{F.2}$$

Taking the logarithm of (F.2) and making use of the following relation

$$K_{2k}(\lambda) = \frac{d}{d\lambda}\frac{1}{2\pi i}\log\left(\frac{\sinh(k\eta - \lambda)}{\sinh(k\eta + \lambda)}\right) \tag{F.3}$$

we have the integral equation for the density of string centers, $\rho(\lambda)$, as follows:

$$\rho(\lambda) = \sum_{j=1}^{\ell} K_{2j-1}(\lambda) - \int_{-\infty}^{\infty}\left(K_{4s}(\lambda - \mu_b) + \sum_{k=1}^{2s-1} 2K_{2k}(\lambda - \mu_b)\right)\rho(\lambda)\, d\lambda. \tag{F.4}$$

For $0 < \zeta \leqslant \pi/m$ we have the following Fourier transform:

$$\int_{-\infty}^{\infty} e^{i\mu\omega} K_m(\mu)\, d\mu = \frac{\sinh((\pi - m\zeta)\omega/2)}{\sinh(\pi\omega/2)}. \tag{F.5}$$

Taking the Fourier transform of (F.4) we have the Fourier transform $\widehat{\rho}(\omega)$ of $\rho(\lambda)$ as follows:

$$\widehat{\rho}(\omega) = \left(\sum_{k=1}^{2s}\widehat{K}_{2k-1}(\omega)\right)\bigg/\left(1 + \widehat{K}_{4s}(\omega) + 2\sum_{k=1}^{2s-1}\widehat{K}_{2k}(\omega)\right) = \frac{1}{2\cosh(\zeta\omega/2)}.$$

Taking the inverse Fourier transform we obtain $\rho(\lambda) = 1/2\zeta\cosh(\pi\lambda/\zeta)$.

Appendix G. Some formulas of the algebraic Bethe-ansatz

Applying the commutation relations between C and D operators we have

$$\langle 0|\prod_{\alpha=1}^{M} C(\lambda_\alpha)\prod_{j=1}^{m} D(\lambda_{M+j}) = \sum_{a_1=1}^{M+1}\sum_{a_2=1;\, a_2\neq a_1}^{M+2}\cdots\sum_{a_m=1;\, a_1\neq a_m,\ldots,a_{m-1}}^{M+m} G_{a_1\cdots a_m}(\lambda_1, \ldots, \lambda_{M+m})\langle 0|\prod_{k=1;\, k\neq a_1,\ldots,a_m}^{M+m} C(\lambda_k)$$

where

$$G_{a_1\cdots a_m}(\lambda_1,\ldots,\lambda_{M+m}) = \prod_{j=1}^{m}\left(d\big(\lambda_{a_j};\{w_j\}_L\big)\frac{\prod_{b=1;b\neq a_1,\ldots,a_{j-1}}^{M+j-1}\sinh(\lambda_{a_j}-\lambda_b+\eta)}{\prod_{b=1;b\neq a_1,\ldots,a_j}^{M+j}\sinh(\lambda_{a_j}-\lambda_b)}\right). \tag{G.1}$$

Let $\{\lambda_k\}_M$ be a set of Bethe roots. We have [4,5]

$$\begin{aligned}&\langle 0|\prod_{k=1;\,k\neq a_1,\ldots,a_m}^{M} C(\lambda_k)\prod_{j=1}^{m}C(w_j)\prod_{\gamma=1}^{M}B(\lambda_\gamma)|0\rangle/\langle 0|\prod_{k=1}^{n}C(\lambda_k)\prod_{\gamma=1}^{M}B(\lambda_\gamma)|0\rangle\\ &= \det\Big(\big(\Phi'\big(\{\lambda_\alpha\}\big)\big)^{-1}\Psi'\big(\{\lambda_\alpha\}\setminus\{\lambda_{a_1},\ldots,\lambda_{a_m}\}\cup\{w_1,\ldots,w_m\}\big)\Big)\\ &\quad\times\prod_{j=1}^{m}\prod_{\alpha=1;\,\alpha\neq a_1,\ldots,a_m}^{M}\frac{\sinh(\lambda_\alpha-w_j+\eta)}{\sinh(\lambda_\alpha-\lambda_{a_j}+\eta)}\prod_{j=1}^{m}\prod_{\alpha=1}^{M}\frac{\sinh(\lambda_\alpha-\lambda_{a_j})}{\sinh(\lambda_\alpha-w_j)}\\ &\quad\times\prod_{1\leqslant j<k\leqslant m}\frac{\sinh(\lambda_{a_j}-\lambda_{a_k})}{\sinh(w_j-w_k)}.\end{aligned} \tag{G.2}$$

References

[1] V.E. Korepin, N.M. Bogoliubov, A.G. Izergin, Quantum Inverse Scattering Method and Correlation Functions, Cambridge University Press, Cambridge, 1993.

[2] N.A. Slavnov, Calculation of scalar products of wave functions and form factors in the framework of the algebraic Bethe ansatz, Theor. Math. Phys. 79 (1989) 502–508.

[3] J.M. Maillet, J. Sanchez de Santos, Drinfel'd twists and algebraic Bethe ansatz, in: M. Semenov-Tian-Shansky (Ed.), Amer. Math. Soc. Transl. Ser. 2, vol. 201, Amer. Math. Soc., Providence, RI, 2000, pp. 137–178.

[4] N. Kitanine, J.M. Maillet, V. Terras, Form factors of the XXZ Heisenberg spin-$1/2$ finite chain, Nucl. Phys. B 554 [FS] (1999) 647–678.

[5] N. Kitanine, J.M. Maillet, V. Terras, Correlation functions of the XXZ Heisenberg spin-$1/2$ chain in a magnetic field, Nucl. Phys. B 567 (2000) 554–582.

[6] N. Kitanine, J.M. Maillet, N.A. Slavnov, V. Terras, On the algebraic Bethe ansatz approach to the correlation functions of the XXZ spin-$1/2$ Heisenberg chain, hep-th/0505006.

[7] M. Jimbo, K. Miki, T. Miwa, A. Nakayashiki, Correlation functions of the XXZ model for $\Delta < -1$, Phys. Lett. A 168 (1992) 256–263.

[8] M. Jimbo, T. Miwa, Algebraic Analysis of Solvable Lattice Models, AMS, Providence, RI, 1995.

[9] M. Jimbo, T. Miwa, Quantum KZ equation with $|q| = 1$ and correlation functions of the XXZ model in the gapless regime, J. Phys. A: Math. Gen. 29 (1996) 2923–2958.

[10] T. Miwa, Y. Takeyama, Determinant formula for the solutions of the quantum Knizhnik–Zamolodchikov equation with $|q| = 1$, Contemp. Math. 248 (1999) 377–393.

[11] F. Göhmann, A. Klümper, A. Seel, Integral representations for correlation functions of the XXZ chain at finite temperature, J. Phys. A: Math. Gen. 37 (2004) 7625–7651.

[12] N. Kitanine, K.K. Kozlowski, J.M. Maillet, N.A. Slavnov, V. Terras, Algebraic Bethe ansatz approach to the asymptotic behavior of correlation functions, J. Stat. Mech. (2009) P04003.

[13] N. Kitanine, Correlation functions of the higher spin XXX chains, J. Phys. A: Math. Gen. 34 (2001) 8151–8169.

[14] O.A. Castro-Alvaredo, J.M. Maillet, Form factors of integrable Heisenberg (higher) spin chains, J. Phys. A: Math. Theor. 40 (2007) 7451–7471.

[15] T. Deguchi, C. Matsui, Form factors of integrable higher-spin XXZ chains and the affine quantum-group symmetry, Nucl. Phys. B 814 (2009) 405–438.

[16] P.P. Kulish, N.Yu. Reshetikhin, E.K. Sklyanin, Yang–Baxter equation and representation theory: I, Lett. Math. Phys. 5 (1981) 393–403.

[17] H.M. Babujian, Exact solution of the one-dimensional isotropic Heisenberg chain with arbitrary spins s, Phys. Lett. A 90 (1982) 479–482.

[18] H.M. Babujian, Exact solution of the isotropic Heisenberg chain with arbitrary spins: Thermodynamics of the model, Nucl. Phys. B 215 (1983) 317–336.
[19] A.B. Zamolodchikov, V.A. Fateev, A model factorized S-matrix and an integrable spin-1 Heisenberg chain, Sov. J. Nucl. Phys. 32 (1980) 298–303.
[20] K. Sogo, Y. Akutsu, T. Abe, New factorized S-matrix and its application to exactly solvable q-state model. I, Prog. Theor. Phys. 70 (1983) 730–738;
K. Sogo, Y. Akutsu, T. Abe, New factorized S-matrix and its application to exactly solvable q-state model. II, Prog. Theor. Phys. 70 (1983) 739–746.
[21] H.M. Babujian, A.M. Tsvelick, Heisenberg magnet with an arbitrary spin and anisotropic chiral field, Nucl. Phys. B 265 (1986) 24–44.
[22] A.N. Kirillov, N.Yu. Reshetikhin, Exact solution of the integrable XXZ Heisenberg model with arbitrary spin. I. The ground state and the excitation spectrum, J. Phys. A: Math. Gen. 20 (1987) 1565–1585.
[23] T. Deguchi, M. Wadati, Y. Akutsu, Exactly solvable models and new link polynomials. V. Yang–Baxter operator and braid-monoid algebra, J. Phys. Soc. Jpn. 57 (1988) 1905–1923.
[24] M. Idzumi, Calculation of correlation functions of the spin-1 XXZ model by vertex operators, Thesis, University of Tokyo, February 1993.
[25] M. Idzumi, Correlation functions of the spin 1 analog of the XXZ model, hep-th/9307129.
[26] M. Idzumi, Level two irreducible representations of $U_q(\widehat{sl_2})$, vertex operators, and their correlations, Int. J. Mod. Phys. A 9 (1994) 4449–4484.
[27] A.H. Bougourzi, R.A. Weston, N-point correlation functions of the spin 1 XXZ model, Nucl. Phys. B 417 (1994) 439–462.
[28] H. Konno, Free-field representation of the quantum affine algebra $U_q(\widehat{sl_2})$ and form factors in the higher-spin XXZ model, Nucl. Phys. B 432 (1994) 457–486.
[29] T. Kojima, H. Konno, R. Weston, The vertex-face correspondence and correlation functions of the eight-vertex model I: The general formalism, Nucl. Phys. B 720 (2005) 348–398.
[30] A. Caradoc, O. Foda, N. Kitanine, Higher spin models with domain wall boundary conditions, J. Stat. Mech. (2006) P03012.
[31] L.A. Takhtajan, The picture of low-lying excitations in the isotropic Heisenberg chain of arbitrary spins, Phys. Lett. A 87 (1982) 479–482.
[32] K. Sogo, Ground state and low-lying excitations in the Heisenberg XXZ chain of arbitrary spin S, Phys. Lett. A 104 (1984) 51–54.
[33] F.C. Alcaraz, M.J. Martins, Conformal invariance and critical exponents of the Takhtajan–Babujian models, J. Phys. A: Math. Gen. 21 (1988) 4397–4413.
[34] I. Affleck, D. Gepner, H.J. Schultz, T. Ziman, Critical behavior of spin-s Heisenberg antiferromagnetic chains: Analytic and numerical results, J. Phys. A: Math. Gen. 22 (1989) 511–529.
[35] B.-D. Dörfel, Finite-size corrections for spin-S Heisenberg chains and conformal properties, J. Phys. A: Math. Gen. 22 (1989) L657–L662.
[36] L.V. Avdeev, The lowest excitations in the spin-s XXX magnet and conformal invariance, J. Phys. A: Math. Gen. 23 (1990) L485–L492.
[37] F.C. Alcaraz, M.J. Martins, Conformal invariance and the operator content of the XXZ model with arbitrary spin, J. Phys. A: Math. Gen. 22 (1989) 1829–1858.
[38] H. Frahm, N.-C. Yu, M. Fowler, The integrable XXZ Heisenberg model with arbitrary spin: Construction of the Hamiltonian, the ground-state configuration and conformal properties, Nucl. Phys. B 336 (1990) 396–434.
[39] H. Frahm, N.-C. Yu, Finite-size effects in the XXZ Heisenberg model with arbitrary spin, J. Phys. A: Math. Gen. 23 (1990) 2115–2132.
[40] H.J. de Vega, F. Woynarovich, Solution of the Bethe ansatz equations with complex roots for finite size: The spin $S \geqslant 1$ isotropic and anisotropic chains, J. Phys. A: Math. Gen. 23 (1990) 1613–1626.
[41] A. Klümper, M.T. Batchelor, An analytic treatment of finite-size corrections in the spin-1 antiferromagnetic XXZ chain, J. Phys. A: Math. Gen. 23 (1990) L189–L195.
[42] A. Klümper, M.T. Batchelor, P.A. Pearce, Central charge of the 6- and 19-vertex models with twisted boundary conditions, J. Phys. A: Math. Gen. 24 (1991) 3111–3133.
[43] J. Suzuki, Spinons in magnetic chains of arbitrary spins at finite temperatures, J. Phys. A: Math. Gen. 32 (1999) 2341–2359.
[44] V.G. Drinfel'd, Quantum groups, Proc. ICM Berkeley, 1986, pp. 798–820.
[45] M. Jimbo, A q-difference analogue of $U(g)$ and the Yang–Baxter equation, Lett. Math. Phys. 10 (1985) 63–69.
[46] M. Jimbo, A q-analogue of $U(gl(N+1))$, Hecke algebra and the Yang–Baxter equation, Lett. Math. Phys. 11 (1986) 247–252.

[47] M. Jimbo, Topics from representations of $U_q(g)$ – An introductory guide to physicists, in: Nankai Lectures on Mathematical Physics, World Scientific, Singapore, 1992, pp. 1–61.

[48] J.M. Maillet, V. Terras, On the quantum inverse scattering problem, Nucl. Phys. B 575 (2000) 627–644.

[49] A. Kuniba, T. Nakanishi, J. Suzuki, Functional relations in solvable lattice models: I. Functional relations and representation theory, Int. J. Mod. Phys. A 9 (1994) 5215–5266;
A. Kuniba, T. Nakanishi, J. Suzuki, Functional relations in solvable lattice models: II. Applications, Int. J. Mod. Phys. A 9 (1994) 5267–5312.

[50] C.S. Melo, M.J. Martins, Algebraic Bethe ansatz for $U(1)$ invariant integrable models: The method and general results, Nucl. Phys. B 806 (2009) 567.

[51] M.J. Martins, C.S. Melo, Algebraic Bethe ansatz for $U(1)$ invariant integrable models: Compact and non-compact applications, Nucl. Phys. B 820 (2009) 620–648.

[52] A. Nishino, T. Deguchi, The $L(sl_2)$ symmetry of the Bazhanov–Stroganov model associated with the superintegrable chiral Potts model, Phys. Lett. A 356 (2006) 366–370.

[53] M. Takahashi, Thermodynamics of One-Dimensional Solvable Models, Cambridge University Press, Cambridge, 1999.

[54] F. Göhmann, N.P. Hasenclever, A. Seel, The finite temperature density matrix and two-point correlations in the antiferromagnetic XXZ chain, J. Stat. Mech. (2005) P10015.